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SCHORLING-CLARK-SMITH
MATHEMATICS SERIES

FIRST-YEAR ALGEBRA

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PREFACE

The study of mathematics is being stressed as never before. Our technical age demands it. The war emergency has increased the demand. Army and navy authorities are especially insistent upon mathematics. They urge every student who has the slightest aptitude for the subject to study as much mathematics as is consistent with his ability. Mathematics is basic to all the technical jobs which are so important today and will be in the future. And at the foundation of mathematics are arithmetic and algebra.

The present emergency does not demand a change in our philosophy of the teaching of mathematics, especially in the early years of the high school, although it does require change in the emphasis of certain topics and the inclusion of material applicable to the needs of our armed forces. The best possible training in algebra for later courses, whether it be further mathematics, or science, or specialized courses in aeronautics, electricity, radio, internal-combustion engineering, is algebra taught with due regard to its fundamental ideas and meanings and adequate consideration for the mastery of skills.

The authors of *First-Year Algebra* believe that instruction in algebra should make it meaningful to the student and should also enable him to acquire a high degree of mastery of its techniques. Accordingly, this text has been designed to develop *both* understanding of principles and mastery of skills. These, the authors believe, should be pursued not as separate objectives but as the two parts of one whole.

Although an understanding of algebra originates and can to a degree be developed by explanation and illustrative examples, it is deepened and fixed by application and practice. And although expertness in manipulation and problem solving comes from abundant practice, understanding of meaning helps greatly in the development of these skills. Meaning helps in use and use helps in meaning; those who understand principles are best in practice, and those who have experience best understand principles. Complete teaching of any subject requires both a presentation of principles and practice in their application. In the judgment of the authors one of the strong features of this text is that to an unusual degree it combines full explanations and developments for meaning and abundant drill for application and manipulation.

Some of the special features of this book are as follows:

A new approach. Algebra is introduced as an extension of arithmetic. The transition from specific relationships in arithmetic to their generalization and thence to their statement in algebraic symbols is a natural development and easy when well handled. This method of procedure gives meaning at once to the letters of algebra. It does away with the false conception that these letters are abbreviations of words, as is often the case when algebra is introduced by means of formulas. Pupils realize that they are working with numerical relationships and not juggling with letters. It has been the experience of the authors and other teachers who have used the text in prepublication form that this building of algebra on the student's previous experience with arithmetic removes much of the abstractness from it; that getting a correct first conception of algebra and its symbolism leads to a far firmer grasp of the subject as a whole. (See Note 1, page 457.)

Full development of concepts and principles. New topics are developed inductively by means of carefully graded questions and illustrations. The effort is directed toward helping the student to develop insight rather than to respond by blindly following a rule which has no adequate meaning for him. There is no substitute for these learning exercises if the principles of algebra are to be understood. Without this careful development of meanings, the study of algebra all too easily resolves itself into a series of mechanical manipulations. (See Note 2, page 458.)

An abundance of drill. As has been stated above, the development of a thorough mastery of algebraic techniques necessitates an abundance of drill material. For all of the important skills, the authors have provided even more drill and problem exercises than many teachers may wish to use. It will be possible to vary the amount of practice according to the needs of each particular class.

Adaptations to different ability levels. All sets of exercises are arranged according to difficulty. The most difficult exercises are separated from the others. Careful choice of exercises makes it possible to adapt the book to groups of varying ability. The chapter reviews enable the teacher to make a diagnosis of the status of the individual student.

Careful teaching of problem solving by a new method of analysis. It is something new for pupils to ask for added work in solving verbal problems. But that is what has happened in classes taught by the method presented here. The confusion, the abstractness in problem solving, is reduced to a minimum by the simple expedient of asking the pupils to assume an answer to a problem and check to see if it is correct before attempting it algebraically. The algebra in problem solving then becomes an extension of the arithmetic with which the pupils are familiar. In trying to check a problem with the assumed answer, pupils become aware of the relationships involved in the problem and can then carry through with comparative ease the same processes with an unknown instead of the assumed answer. (See Note 9, page 460.)

Simple, direct style. A lucid, flowing, readable style is a great advantage even in a mathematical text, and the authors have therefore attempted to gear the presentation to the mental maturity level of the student of first-year algebra. They have tried to be concise, but not so concise as to hinder comprehension. Note the numerous special introductions and summaries serving to orient the pupil in his progress through the course. Pupils are shown the *why* as well as the *what* in algebra.

Modern appearance and content. The pictures and many of the exercises are designed to show the use of algebra in the modern world of aviation and industry.

Notes to the Teacher. In order to help the teacher to understand the purpose of the authors, several pages of Notes to the Teacher are given at the end of the book.

The authors are deeply indebted to the teachers of algebra who have read, used, and criticized the manuscript. Many of their suggestions have been incorporated herein.

R. S.
J. R. C.
R. R. S.

CONTENTS

Old and new in transportation and computation.

INTRODUCTORY

Algebra and Its Place in the Modern World

"The human mind has never invented a labor-saving machine equal to algebra." *The Nation*

Algebra grows out of arithmetic, but it goes far beyond arithmetic. Because of the new kinds of numbers it uses and the new methods it employs, algebra can do many things that arithmetic cannot do. Before starting on a study of the processes and methods of this great branch of mathematics, a few general facts relating to it will be of help to you.

In algebra much use is made of symbols. In arithmetic you used the figures 1, 2, 3, 4, and so on, to represent numbers. In algebra, in addition to these figures, letters also are employed as symbols for the numbers that are used. These letters are symbols for numbers just as the figures of arithmetic are. Also, in algebra we have other symbols that indicate the relationships of numbers and the operations to be performed, just as in arithmetic the signs of operation ($+$, $-$, $\times$, $\div$) tell you when to add, subtract, multiply, and divide. The whole subject of algebra is essentially a symbolism, or set of symbols, that make up a language to express mathematical ideas.

How important symbols may be is shown by the way we use the little marks we call letters and words when we read and write. These marks are language symbols, and they not only help us to become acquainted with what others have learned but also help us in our own thinking. So in algebra we have a set of symbols that form a second language, understood by the peoples of all nations, in which numerical facts and ideas are conveyed to us in a brief manner and in easily understood form. It is in this language that the quantitative relationships of the things in the world about us are explained and discussed. It is important for you to understand that in taking up algebra you are beginning the study of a language and that you must learn to read this language if you are to comprehend the ideas that are expressed in it.

With algebra it is possible to work with variable numbers. In arithmetic each figure stands for a particular number. The number never varies but has a particular value, no more and no less. In algebra a letter may stand for *any number*, no matter what value it may have. This enables us to make general statements about numbers that will be true for all numbers. Algebra has often been called *generalized arithmetic*, an arithmetic that gives the answers to problems, no matter what the values of the numbers in the problems may be.

The importance of this ability to work with variable numbers is beyond the power of words to express, for we live in a varying world where things depend on one another. The temperature of a summer day may depend on whether the day is clear or cloudy or on the direction of the wind. The force with which a heavy body hits the ground depends on the distance it falls. The speed with which an airplane will rise depends on many factors. Everything about us is relative, one thing depending on another.

When we wish to deal with this shifting, varying world in terms of numbers, we must have algebra. Arithmetic with its particular numbers might solve the problems of a changeless, static world or of some particular moment of a moving one, but only algebra with its general numbers can deal with a world of dependence and constant change. Again and again as your course proceeds, you will meet problems in which the numbers are variables, the value of one depending on the value of another. Note how helpless we should be before such problems if we brought to their solution only arithmetic and the particular numbers with which it works.

In the equation and in the rules for its solution, algebra has developed a wonderful system of reasoning. Algebra has been defined as "the science of the equation," and it has been described as "a way of thinking." In algebra all the known facts relating to a problem are arranged in order and set in the form of an equation. Then step by step, following the known laws of equation handling, the equation is solved. By this system of algebraic reasoning, conclusions far beyond the power of the unaided human mind can be reached.

It was by the use of the equation that Sir Isaac Newton discovered the law of gravitation and Einstein developed his theory of relativity. It is by the use of the equation that engineers calculate the stresses in great bridges and in airplane wings, operations far too complex to be carried out in any other way. Of the equation Sir Oliver Lodge has said, "An equation is the most serious and important thing in mathematics." The very heart of algebra and of all higher mathematics lies in the equation and its use.

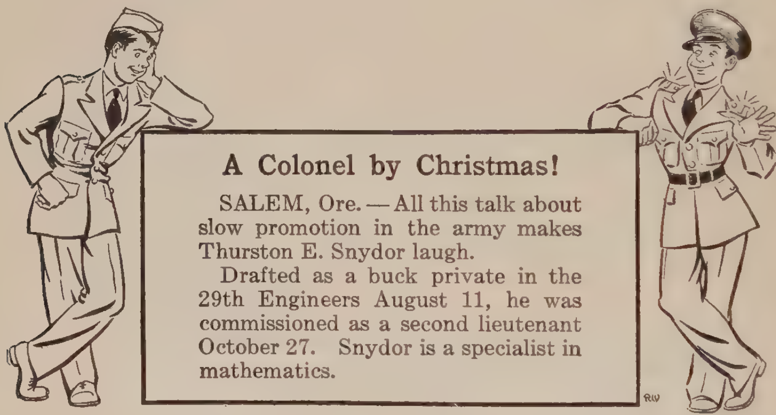
Don Guy

The great wind-driven electric plant that stands on a mountain top near Castleton, Vermont. The amount of electricity it generates depends on the velocity of the wind. Any general statement about its output must be made in terms of the variable numbers of algebra.

In the first three chapters of this book the basic ideas and fundamental processes of algebra are developed, and these chapters will give you a conception of what algebra is and how it can be used. After completing each of these, turn back and read again these general introductory statements about algebra. It is important that you understand in a wide general sense the particular things you are doing, and what has been here said will have more and more meaning for you as you progress in your course.

Why do we study algebra? Why has this ancient Oriental subject been given so prominent a place in our schools? Only because of its great usefulness to us. It is stating a simple truth to say that without algebra modern life would not be possible.

Without algebra we could not determine how strong we must make the frameworks of great buildings and long bridges. We should not be able to construct high dams and design generators that turn the energy of falling water into electric power. Without it we should know little of aviation, telescopes, microscopes, radios, or insurance; of electrons that swirl in their tiny orbits about the nucleus of an atom or of the distances to the stars. Algebra has made possible the development of all our fast-moving and complicated machines. It is the universal tool of scientists. Like the air that sustains our lives, the benefits that have come from its use surround and envelop us and are freely shared by us all.



A Colonel by Christmas!

SALEM, Ore.—All this talk about slow promotion in the army makes Thurston E. Snydor laugh.

Drafted as a buck private in the 29th Engineers August 11, he was commissioned as a second lieutenant October 27. Snydor is a specialist in mathematics.

CHAPTER I

GENERAL ARITHMETIC

Algebra is a generalized arithmetic. This chapter will make clear what is meant by that statement by starting you to think more generally about number relationships with which you are familiar and showing you how to express these general relationships by means of letters which are thought of as numbers.

We shall make our study by using the numbers of algebra in actual computations of various kinds, learning the algebraic methods of operation as we have need for them. Before we begin our work with these numbers, let us make sure that we have a clear understanding of the nature of symbolism and what the letter symbols of algebra really mean.

Nature of Written Symbols[1]

What do you see written on the black square at the right? You say, "cat." But you do not see the animal. You see a word that makes you think of the animal. If you had not learned to read written language symbols, you would have to say, "I see some marks." The marks then would have no meaning at all; they are senseless symbols until you have learned what they mean.

You have undoubtedly seen something written in symbols that mean nothing to you — a Chinese laundry check, a Hebrew Bible, an inscription from an Egyptian tomb, or a Koran written in Arabic. But those same symbols have meaning to anyone who has learned to understand them. So it is with

[1] TO THE TEACHER. See Note 1 on page 457.

all written symbols. They are merely marks invented to convey ideas. They have no meaning at all of themselves. *The meaning of any symbolism has to be learned through study and practice before we can understand and use it.* Whether it be the symbolism of language, mathematics, or signal codes, this is true of it.

Here are some algebraic symbols taken from a book on aeronautics.

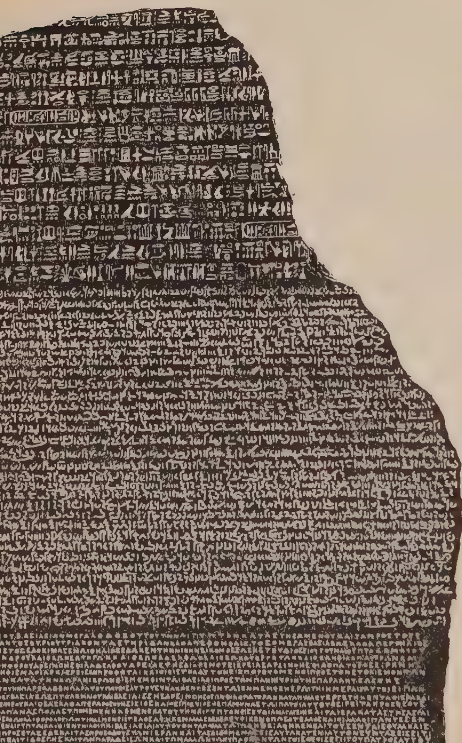
Coefficient of lift, $C_L = \frac{L}{\frac{\rho}{2}SV^2}$

Lift, $L = C_L\frac{\rho}{2}SV^2$

Coefficient of drag, $C_D = \frac{D}{\frac{\rho}{2}SV^2}$

Drag, $D = C_D\frac{\rho}{2}SV^2$

These symbols look strange, do they not? You probably cannot read them. But this symbolism is not confusing to one who knows its meaning. On the contrary, by use of the algebraic symbolism these important facts about aeronautics are expressed much more briefly and clearly than by other means. And the symbolism has the same meaning for all good aviators, regardless of the language they speak — provided they know their algebra as they should.



This is a portion of the Rosetta Stone, which gave the world the key to the symbols of the ancient Egyptian language. The inscription is a decree written in two forms of Egyptian and in Greek. Until the Rosetta Stone was found in 1799 and studied, Egyptian hieroglyphics had no meaning for scholars of the modern world.

A Letter for "Any Number" †

The letters of algebra, such as a or b or x or y, are a new symbolism to you. As you work with them, hold in mind that they are not abbreviations, but symbols for numbers. They are to be operated with just as the numbers of arithmetic are.

(1) How much is 6×0? 8×0? 15×0? (As you read, try to answer all questions before looking at the answers that follow them. For this Ex. (1) the answer in each case is 0.)

(2) Give other examples of the truth illustrated in Ex. (1).

(3) Can you in a single sentence state the general truth you have illustrated? (Any number times zero is zero.)

Look at the two following statements:

Any number times zero is zero.

$$a \times 0 = 0$$

The second statement is the same as the first, but is written in mathematical symbols. Say the first statement to yourself and at the same time look at the second. When a mathematician reads the second statement, he *thinks* the first statement. When he sees a letter a or b or c — any letter of the alphabet — in a statement like this one, he does not think of it as a letter; he thinks of the letter as meaning "any number." The statement above could just as well be written, $n \times 0 = 0$.

In algebra, letters represent numbers.

One benefit of this symbolism of algebra is that by means of it you can state general truths about arithmetic very briefly.

(4) What does the a mean in $a \times 0 = 0$? (Any number.) Could the number be 3? Could it be 5? 1042? (Yes.) Name some other numbers that a can be.

(5) Read $b \times 0 = 0$ as a sentence in English. (Any number times zero is zero.)

When you see a letter in algebra, do not think of it as a letter; think of it as a number.

† To the Teacher. See Note 2 on page 458. The exercises of the text indicated by a number in parentheses are intended for class discussion.

Now that you have had this explanation of the meaning of a letter in algebra, try to answer the following questions before looking at the answers that are given in parentheses.

(6) What does the statement, $1 \times a = a$, mean? Do you believe the statement is true? Give several examples to illustrate the statement.

(It means: One times any number is that number. For example, $1 \times 3 = 3$, $1 \times 9 = 9$, $1 \times 54 = 54$.)

Did you make the mistake of thinking at first that the statement, $1 \times a = a$, means "One times any number is any number"? That, of course, is not true. 1×8 is not 10. The fact that we use a both times indicates that whatever number we think of for the first a must be the number for the second a.

(7) What does this statement mean: $a + b = b + a$? Do you believe it is true? Give several illustrative examples. (It means that if you have two numbers and add the first number to the second you will get the same sum as if you had added the second number to the first. For example, $5 + 4 = 4 + 5$ or $62 + 75 = 75 + 62$.)

(8) What does a mean in Ex. (7)? (Any number.) What does b mean? (Any number.)

(9) In Ex. (7), if you choose one of the a's as 4, what number does the other a represent? (4; whatever number we choose for one a must be the number for the other a.)

(10) Does $a + b + c = c + b + a$? (Yes. $5 + 4 + 2 = 2 + 4 + 5$.)

Note that you make use of the principle we have been discussing in Exs. (7) to (10) when you check addition. You first add a column *up* to get the sum and then you add it *down* to check the sum. That is, $a + b + c + d$ must equal $d + c + b + a$.

(11) Does $a - b = b - a$? (No; 6 from 10 is not the same as 10 from 6.)

The statement $\boldsymbol{a + b = b + a}$ is a fundamental law of algebra. It is called the ***commutative law of addition.*** Explain what it means.

Letters that represent numbers are called ***literal numbers*** to distinguish them from the ***arithmetic numbers*** (a-rith-met'ic) which you use in arithmetic (a-rith'me-tic). The word "literal" comes from the Latin *litera* meaning letter.

Numbers that are to be added are called ***addends.***

Exercises

In these exercises give a numerical example of each statement:

1. $a + b = b + a$
2. $d \times 0 = 0$
3. $1 \times f = f$
4. $a + b + c = c + a + b$
5. $a \times b = b \times a$
6. $a \times b \times c = c \times a \times b$
7. $a \div a = 1$
8. $a - a = 0$

9.[1] If $a + b = c$, then $c - b = a$

10. If $a \div b = c$, then $c \times b = a$

11. If $a - b = c$, then $c + b = a$

12. The sum of a and b is the same as the sum of b and a.

13. The product of a and b is the same as the product of b and a.

14. If the sum of a and b is divided by 2, the result is the same as the sum of one half of a and one half of b.

15. If I add a and b and then multiply the sum by 2, the result is the same as if I multiply a and b separately by 2 and then add.

16. In the statement $a - b = c$, a can be any number. Can b be any number? (If a is 5, can b be 8?) Is c any number?

The statements in Exs. 14–15 may be made very simply by means of algebraic symbolism which you will learn later. Just as a matter of interest, they are: Ex. 14, $\frac{a+b}{2} = \frac{1}{2}a + \frac{1}{2}b$; Ex. 15, $2(a + b) = 2a + 2b$

[1] The line above Ex. 9 and in other lists of exercises separates the easier from the more difficult exercises.

Algebraic Symbols in Multiplication †

Multiplication is a fundamental operation in algebra as it is in arithmetic, but in algebra a different symbolism may be employed, as the following study will make clear.

(1) Can you think of a law in multiplication similar to $a + b = b + a$? If you can, give a numerical example to illustrate it. ($5 \times 4 = 4 \times 5$.)

(2) Is this true of any two numbers, or is it true only of the numbers you have given? (It is true of any two numbers.)

(3) State the law in words. (One number times a second number gives the same product as the second times the first. In both addition and multiplication the order of the numbers makes no difference.)

(4) How would you state the law, using a and b to represent any two numbers? ($a \times b = b \times a$.)

In algebra the times sign ($\times$) *is usually omitted between an arithmetic number and a literal number or between two literal numbers.* Thus $3 \times a$ is written $3\,a$ and $a \times b$ is written ab. The above law is written $ab = ba$.

There are two other ways of indicating multiplication. The raised dot between two numbers means multiplication. Thus $3 \cdot 2$ means 3×2. And $(3)(2)$ also means 3×2.

(5) Give several examples of the law $ab = ba$.

(6) What does the a represent in the above statement? (Any number.) What does the b represent? (Any number.) Could a be 7 and b be 3? (Yes.)

The statement $ab = ba$ is another fundamental law of algebra. It is called the *commutative law of multiplication.*

(7) What is the value of ab when $a = 5$ and $b = 3$? (15.) when $a = 6$ and $b = 4$? (24.)

(8) What is the value of ba when $a = 5$ and $b = 3$? (15.) when $a = 6$ and $b = 4$? (24.)

† To the Teacher. See Note 2 on page 458.

(9) What is the value of mn when $m = 4$ and $n = 5$? (20.) What is the value of nm? (20.)

(10) What does $2b$ mean? ($2 \times b$ or 2 times any number for which b stands.) What is the value of $2b$ when b is 6? (12.)

(11) What is the value of $2c + 5$, if $c = 4$? ($2 \times 4 + 5 = 13$.)

(12) Multiply as indicated: $3 \cdot 5$; $6 \cdot 7$; $(9)(8)$; $(7)(5)$.

The 5 in the expression $5a$ is called the *numerical coefficient* of a. The 3 in $3a$ and the 7 in $7b$ are numerical coefficients.

Exercises

1. In the expression $4a + 5b + 7c$, what is the numerical coefficient of a? of b? of c?

What is the value of the following when $a = 5$ and $b = 2$?

2. $a + b$
3. $a + a$
4. $a + a + a$
5. $a - b$
6. $a - a$
7. $b - b$
8. ab
9. $2a$
10. $3a$
11. $5a$
12. $64a$
13. $7b$
14. $72b$
15. $a - b - b$
16. $2ab$
17. $3ab$
18. $4ab$
19. $5ab$
20. $a + 3$
21. $a + 5$
22. $a + 7$
23. $a + 9$
24. $a - 4$
25. $a - 3$
26. $a - 1$
27. $b - 1$
28. $b + 22$
29. $b + 124$
30. $2a + 5$
31. $2a + 7$
32. $2a + 1$
33. $2a + a$
34. $2a + 3a$
35. $2a + 5a$
36. $2a + 3$
37. $2a + 3b$
38. $2a + 15$
39. $2a + 5b$
40. $2a + 17$
41. $2a + 7b$
42. $3a - 1$
43. $3a - b$
44. $3a - 2b$
45. $3a - 2$
46. $2a + 3b + 5$
47. $2a + 3b - 5$
48. $a + 0$
49. $b \times 0$
50. $ab \times 0$
51. $2a + 3b + 0$
52. $\frac{1}{2}a + \frac{1}{3}b$

Perform the indicated operations:

53. $9 \cdot 5$	**59.** $(\frac{1}{2})(\frac{1}{2})$	**65.** $(8)(1.25)$
54. $8 \cdot 7$	**60.** $(\frac{1}{2})(\frac{1}{3})$	**66.** $(.5)(.5)(.5)$
55. $3\frac{1}{2} \cdot 3\frac{2}{3}$	**61.** $(\frac{1}{2})(\frac{1}{2})(\frac{1}{2})$	**67.** $(.1)(.1)$
56. $2 \cdot 2 \cdot 2$	**62.** $(2)(3)(4)$	**68.** $(.1)(.003)$
57. $3 \cdot 3 \cdot 3 \cdot 3$	**63.** $(4)(0)(5)$	**69.** $(2.35)(2.35)$
58. $9\frac{3}{4} \cdot 6\frac{1}{3}$	**64.** $(9)(8)(3\frac{1}{2})$	**70.** $(982)(.01)$

Find the value of the following when $a = \frac{1}{2}$, $b = \frac{2}{3}$, *and* $c = \frac{3}{4}$:

71. $a + b$	**74.** $a + b + c$	**77.** $3a + 5b - 2c$
72. ab	**75.** $2a + 3b$	**78.** abc
73. $a + a$	**76.** $3a - 2b$	**79.** $5ac$

Find the value when $a = 5$, $b = .23$, *and* $c = 1.5$:

80. $a + b$	**82.** ab	**84.** $2b + 3$
81. $a - b$	**83.** abc	**85.** $3 - 2b$

Meaning of Multiplication†

You know that $7 \times 9 = 63$. You have probably used that multiplication fact many times since you first learned to multiply numbers. Do you know what 7×9 means? It means that if you add 9 seven times, the sum is 63. Multiplication is merely a simpler way of getting the sum of any number of equal addends.

342
342
342
342
342

(1) What is a simple way to find the sum of the numbers at the right? (Multiply 342 by 5.)

(2) Add the following columns by means of multiplication:

7	8	9	13	242	a
7	8	9	13	242	a
7	8	9	13	242	a
7	8	9	13	242	a

(The first sum is 4×7, or 28; the second is 4×8, or 32; the last is $4 \times a$, which is written $4a$.)

(3) What does the statement $a + a + a + a = 4\,a$ mean? (Any number added 4 times is the same as 4 times that number.)

(4) $7 + 7 + 7$ is 3×7; $9 + 9 + 9$ is 3×9; and $259 + 259 + 259$ is 3×259. Is this true because of the particular numbers chosen or is it true of any number? (True of any number.)

(5) Give numerical examples of the following statements: $a + a = 2\,a$; $a + a + a = 3\,a$; $a + a + a + a + a + a + a = 7\,a$.

(6) Write the following more simply: $b + b + b$; $c + c + c + c + c$. ($3\,b$, $5\,c$.)

(7) Give a numerical example illustrating the fact that $5\,a + 3\,a = 8\,a$. ($5 \times 342 = 342 + 342 + 342 + 342 + 342$ and $3 \times 342 = 342 + 342 + 342$. Both sums together are 8×342.)

(8) Give a numerical example of the fact that $5a - 3a = 2a$. ($5 \times 697 = 697 + 697 + 697 + 697 + 697$. If now we cross off — i.e., subtract — three of these addends, we have two left.)

(9) Complete the following statements:

$7\,a + 5\,a = \underline{\ ?\ }\,a$	$7\,b + 2\,b - 3\,b = \underline{\ ?\ }$
$9\,n - 3\,n = \underline{\ ?\ }\,n$	$8\,p - 4\,p + 9\,p = \underline{\ ?\ }$
$3\,a + a = \underline{\ ?\ }$	$8\,a + 3\,a + a = \underline{\ ?\ }$
$15\,a - a = \underline{\ ?\ }$	$6\,n - 4\,n + n = \underline{\ ?\ }$

Note that a and $1\,a$ mean the same thing; the 1 is usually omitted.

(The answer to the first statement above is $12\,a$; to the second, $6\,n$; and to the last, $3\,n$.)

In the expression $7\,b + 2\,b - 3\,b$, the $7\,b$, $2\,b$, and $3\,b$ are *terms*. Terms are separated by plus and minus signs. Adding and subtracting terms is called *combining terms*.

(10) Combine terms in the following:

$2\,a + 3\,a$	$5\,b - 2\,b$	$7\,b - 4\,b$
$5\,a + 7\,a$	$6\,a + a$	$8\,c - 5\,c + 3\,c$

(The answer to the first is $5\,a$; to the second, $12\,a$; to the last, $6\,c$.)

Exercises

Combine terms in the following:

1. $a + a$
2. $b + b + b$
3. $5a + 7a$
4. $9a - 4a$
5. $62a - 45a$
6. $11a + 2a - 12a$
7. $5a + a$
8. $7x - x$
9. $10n - 4n - n$
10. $8b - 2b$
11. $9a + 5a$
12. $8m - 2m$
13. $3p + 2p - 5p$
14. $3a + a + 6a$
15. $b + b$
16. $4b + 2b - 3b$
17. $4x + 6x - x$
18. $9n - 3n + 6n$
19. $6b - b - 3b$
20. $4d + d + d$
21. $4c + 2c - 3c$
22. $8y - 8y$

23. In the expression $7a + 10b + 8c$, how many terms are there? What are they? What are the numerical coefficients?

Combine terms:

24. $\frac{1}{2}a + \frac{1}{2}a$
25. $\frac{1}{3}a + \frac{1}{3}a$
26. $\frac{1}{4}a + \frac{1}{4}a + \frac{1}{4}a$
27. $\frac{2}{3}a + \frac{2}{3}a$
28. $\frac{1}{2}a + \frac{1}{3}a$
29. $\frac{1}{3}a + \frac{1}{4}a$
30. $\frac{2}{3}m + \frac{3}{4}m$
31. $\frac{2}{3}x + \frac{3}{4}x + x - 2x$
32. $6.4x + 3.2x$
33. $6x + 4.5x$
34. $8.6m - 2.8m$
35. $10.36r - 6.2r$
36. $1.8a + 3.5a + 6.7a + 8.235a - 7.28a$
37. $3.5b + 2.25b + 6.73b - 5.4b$
38. $\frac{1}{2}x + \frac{1}{4}x + \frac{1}{8}x + \frac{5}{4}x + x$
39. $c + 0.1c$
40. $x + 0.03x$
41. $54s - 2.75s$

42. If a and b are numbers, is ab a number? What is the result of combining terms in the following?

(*a*) $7ab + 3ab$ (*b*) $9ab - 6ab$ (*c*) $5abc + 9abc$

Adding Literal Numbers

Terms such as $5\,b$ and $3\,b$, or $4\,a$ and a, or $3\,n$ and $2\,n$, or $6\,ab$ and $9\,ab$ are called ***like terms;*** the two terms of each pair are alike in their literal parts. Terms that are unlike in their literal parts, such as $3\,a$ and $5\,b$, are called ***unlike terms.*** We cannot combine unlike terms as we do like terms. Addition and subtraction of unlike terms can only be indicated.

(1) You know how to combine terms in the expression $9\,a - 3\,a + 2\,a$. Would you know how to write the expression $6\,a + 9 - 2\,a + 6$ more briefly? Try it. Then check your answer by reading further.

(2) Is $n + 2$ the same as $n + 2\,n$? If you are not sure, let $n = 5$ and find the value of both $n + 2$ and $n + 2\,n$. Do you get the same result for both?

(3) The expression $5\,b + 3$ means 3 more than $5\,b$; $5\,b + 3\,b$ means $3\,b$ more than $5\,b$. Let $b = 2$. Do you get the same result for $5\,b + 3$ and $5\,b + 3\,b$?

(4) The expression $5\,b + 3\,b$ may be written more simply as $8\,b$, but $5\,b + 3$ cannot be written more simply unless you know the value of b.

(5) Simplify $4\,a + 2 - a + 3$. The answer is $3\,a + 5$. Be sure you know how this answer was found.

(6) Now you should be able to write $6\,a + 9 - 2\,a + 6$ in a briefer way.

When you are given an algebraic expression and asked to combine terms, you can combine only those terms that are alike in their literal parts — that is, like terms. Arithmetic numbers may be considered as like terms.

Like terms are combined by adding or subtracting the numerical coefficients, as the signs indicate.

Exercises

Simplify by combining like terms whenever possible:

1. $n + 2\,n$

2. $n + 2$

3. $3\,a + 2\,a + 5\,a$

4. $3\,a + 2 + 5\,a$

5. $3 + 2a + 5$

6. $7b - 2b + 3b$

7. $7b + 3 - 2$

8. $7 + 3b - 2b$

9. $n + 3n + n + 3$

10. $n + 3 + n + 3$

11. $n + 3 + 1 + 3n$

12. $2a + 3b - a + 7b$

13. $3a + 7a - 4a$

14. $5b - b + 2x$

15. $9m + 5n - 4m$

16. $3x + 2y + 5x + 7x$

17. $3c + 2d + 5c$

18. $3c + 2c + 5d$

19. $b + 4c + b - 3$

20. $5m + m - 6 + 2n$

21. $a + \frac{1}{2}a - 3$

22. $\frac{2}{3}b + \frac{5}{2}b + 2c$

23. $2x + .25x + .05y$

24. $c + .05c$

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25. $n + .12n$

26. $n + .12$

27. $3n - 1.02n + 5.43$

28. $\frac{1}{2}a + \frac{1}{4}a + \frac{3}{8}a$

29. $2b + 2.5b - 0.1b$

30. $4.5a + 1.7b$

31. $7\frac{3}{4}a + 5\frac{1}{4}b + 9\frac{1}{2}a$

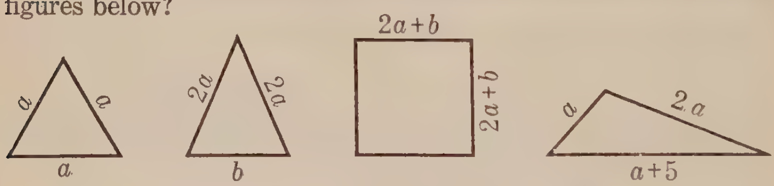
32. $8\frac{3}{8}x - 4\frac{3}{4}x$

33. $b + .75a - b + 1.25a$

34. $9ab + 3bc + 7cd$

Using Algebraic Symbolism[1]

1. What is the perimeter of (distance around) each of the figures below?



2. If it is a miles from Boston to Springfield and b miles from Springfield to New York, how far is it from Boston through Springfield to New York?

3. Profit equals the selling price minus the cost minus the expense of doing business. If a man sells a dollars' worth of goods which cost b dollars and the cost of doing business is c dollars, what is the profit?

[1] To the Teacher. See Note 3 on page 458.

4. How many minutes are there in a hours?

5. A certain window is twice as high as it is wide. If it is a feet wide, how high is it? If it is b feet high, how wide is it?

6. A pilot flew a miles on Monday, twice as far on Tuesday, and 250 more miles on Wednesday than on the other two days combined. How far did he fly in all?

7. The length of a rectangle is $2a + 3$ inches and the width is $a - 3$ inches. What is the perimeter?

8. The area of a rectangle is found by multiplying the number of units in the length by the number of units in the width. If the length is l inches and the width is w inches, what is the area?

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9. In the stock market reports in the newspaper a rise in the price of a stock is indicated by a plus sign and a loss is indicated by a minus sign. If the gains and losses in a certain stock over a six-day period were $+\frac{1}{2}p$, $-\frac{1}{8}p$, $+p$, $+\frac{1}{4}p$, $-\frac{3}{4}p$, and $-2p$, was the total result a gain or a loss? How much?

10. If you were told that the perimeter of the last figure in Ex. 1 is 21 inches, could you find what the value of a is? What is it?

Algebraic Symbolism in Subtraction†

To learn to think of letters as representing numbers, experience with them in various number relationships is required. The exercises that follow in checking subtractions and divisions will give you further practice with literal numbers.

Subtract
15
8
7

(1) How do you check the example at the right? (Add 7 and 8 to see if the sum is 15.)

(2) Does this method work because the numbers are 15 and 8 or will it work for any two numbers? (For any two numbers.)

(3) What is the general rule for checking subtraction? (Add the *remainder* to the *subtrahend* to see if the sum is equal to the *minuend*.)

(4) In the example at the right, what does a represent? (Any number.) What does b represent? (Any number equal to or less than a. Later you will learn how it can represent any number in an example like this.) What does c represent? (The remainder. It represents only one number, the result of the subtraction.) Another way to write this example is $a - b = c$.

Subtract
a
b
c

(5) In Ex. (4) what do you know about $c + b$? ($c + b = a$.) Give several numerical examples to illustrate this fact. (See the subtraction example in Ex. (1).)

(6) Complete: If $a - b = c$, then $c + b =$ _?_ ($c + b = a$.)

Algebraic Symbolism in Division†

Division is expressed in several ways. Early in arithmetic you learned that $8\overline{)24}$ means 24 divided by 8. You can write the same thing as $24 \div 8$. In algebra, the most common way to indicate division is by the fraction sign, or horizontal line. $\frac{24}{8}$ means $24 \div 8$; $\frac{m}{n}$ means $m \div n$.

(1) Write each of the following, using the fraction sign:

$16 \div 4 \qquad 56 \div 13 \qquad 2 \div 5 \qquad a \div b$

(2) Read each of the following, translating the fraction sign as "divided by":

$$\frac{20}{5} \qquad \frac{103}{26} \qquad \frac{4}{5} \qquad \frac{x}{y}$$

(3) Find the value of each of the following when $a = 5$ and $b = 3$:

$$\frac{a}{b} \qquad \frac{b}{a} \qquad \frac{a+b}{2} \qquad \frac{a-b}{3} \qquad \frac{5}{a+b}$$

(4) How do you check the example at the right? (Multiply 3 by 6 to see if the product is 18.)

$$6\overline{)18}\ \ 3$$

(5) Does this method work because the numbers are 18 and 6 or will it work for any two numbers? (For any two numbers.)

(6) What is the general rule for checking division that "comes out even"? (Multiply the *quotient* by the *divisor* to see if the product is equal to the *dividend*.)

(7) If a and b are any two numbers and I do the example at the right and the quotient is c, how would I check the example? (I would multiply c by b to see if I get a.)

$$b\overline{)a}\ \text{quotient } c$$

(8) $\frac{24}{3} = 8$. What is the check? ($8 \times 3 = 24$.)

(9) If $\frac{a}{b} = c$, how much is bc? (a.)

Exercises

Write the following, using the fraction sign to indicate division:

1. $42 \div 19$
2. $19 \div 42$
3. $7 \div 3$
4. $3 \div 7$
5. $12 \div 6$
6. $6 \div 12$
7. $a \div b$
8. $b \div a$
9. $2m \div 3n$
10. $3n \div 2n$
11. $p \div 5$
12. $5 \div p$

State what the following mean, translating the fraction sign as "divided by":

13. $\frac{19}{5}$
14. $\frac{3}{5}$
15. $\frac{bh}{2}$
16. $\frac{a-b}{3}$
17. $\frac{5a}{4}$
18. $\frac{6}{a+b}$

Find the value of each of the following when $a = 7$ and $b = 3$:

19. $\frac{a}{b}$
20. $\frac{b}{a}$
21. $\frac{2a}{3}$
22. $\frac{3}{2a}$
23. $\frac{5b}{6}$
24. $\frac{7b}{4}$
25. $\frac{a+b}{b}$
26. $\frac{a-b}{3}$
27. $\frac{2a+3b}{5}$
28. $\frac{a}{2} + 3$
29. $\frac{a+3}{2}$
30. $\frac{a}{2} + \frac{b}{4}$
31. $\frac{2a}{3} + \frac{5}{6}$
32. $\frac{2a-4b}{5}$
33. $\frac{ab}{5}$
34. $\frac{2ab}{7}$
35. $\frac{3ab-5b}{6}$
36. $\frac{a}{b} - \frac{b}{a}$

Write the following as simply as you can, using algebraic symbols:

37. Two times a
38. Five times c
39. a times b
40. The sum of a and b
41. The difference between a and b (a being larger than b)
42. Five more than a
43. Six less than b
44. a less than b
45. Five less than two times a
46. Seven more than three times b
47. The product of a and b
48. a divided by b
49. a divided by the sum of a and b
50. Twice the product of x and y
51. Five times the product of c and d

Find the value of the following when $a = 2\frac{1}{2}$, $b = 3\frac{1}{2}$, *and* $c = 3\frac{1}{4}$:

52. $5ab + 2c$
53. $7ab - 2ac$
54. $5abc + bc + ac$
55. $ab + ac + 2bc$
56. $\frac{a}{b}$
57. $\frac{b}{a}$
58. $\frac{a+b}{ab}$
59. $\frac{ab}{a+b}$
60. $\frac{a}{b} + \frac{b}{a}$
61. $\frac{1}{a} + \frac{1}{b}$
62. $\frac{abc}{a+b+c}$

63. Is $2a + 3a$ more or less than $2a + 3$? (Try numbers greater than 1 and less than 1 as well as 1 itself.)

A Letter for "Some Number" †

In our study thus far, the letters have for the most part meant "any number." Sometimes a letter represents "some number" or "some numbers." For example, $n + 4 = 10$ means "four added to *some* number is ten." It is obvious that you cannot add 4 to *any* number and get 10. The problem is to find the number for which n stands in such statements.

In the next chapter you will learn definite methods of finding this *unknown number*. First you need to understand what such statements mean, so that you can be sure that you are finding the right number. All the numbers in the following statements are easy to discover by trial, if you see no other way to find them.

(1) Read this statement, using "some number" for the letter: $n + 5 = 8$. (5 added to some number gives 8.)

(2) Is 4 the number? (No, because $4 + 5$ is not equal to 8.)

(3) Is 3 the right number? (Yes, because $3 + 5$ is 8.)

(4) Read this statement, using "some number" for the letter: $n - 2 = 3$. (2 subtracted from some number is 3.)

(5) Is the number 8? (No, because $8 - 2$ is not 3.)

(6) Is it 5? (Yes, because $5 - 2 = 3$.)

The algebraic statements you have been working with are called equations. An *equation* is a mathematical statement that one number expression equals another. In an equation everything to the left of the equal sign is called the *left side* or the *left member;* everything to the right of the equal sign is called the *right side* or the *right member.* The letter that stands for "some number" in these equations is called the *unknown number* or simply *the unknown.*

(7) Read this equation: $2n + 3 = 7$. (2 times some number plus 3 is 7.)

(8) Is the unknown number 3? (No; $2 \times 3 + 3$ is 9, not 7.)

(9) Is it 2? (Yes, because $2 \times 2 + 3 = 7$.)

Finding whether an answer is correct is called *checking* the answer. You have done this by *substituting* the number you guessed for the unknown in the given equation to see if the left member was then equal to the right.

(10) Read the following equations, then find what the unknown number is. Be sure you are correct by *checking* as you have done in the previous equations.

$2n = 8$	$3n = 21$	$2n + 1 = 7$
$\frac{a}{2} = 5$	$\frac{a}{3} = 9$	$2n - 3 = 7$

Exercises[1]

State in words what each of the equations in Exs. 1–16 means. Find the value of the unknown by trial, and check your answers. Use the form shown below.

EXAMPLE. $3n + 5 = 11$

Three times a number plus 5 equals 11

Then $n = 2$

CHECK. $3 \times 2 + 5 = 11$

$6 + 5 = 11$ and $11 = 11$

1. $2n = 6$
2. $n - 2 = 6$
3. $\frac{n}{2} = 6$
4. $n + 2 = 6$
5. $3n = 9$
6. $\frac{n}{3} = 9$
7. $n + 3 = 9$
8. $n - 3 = 9$
9. $2a + 1 = 5$
10. $2a - 1 = 5$
11. $3b + 2 = 11$
12. $3b - 2 = 7$
13. $\frac{a}{2} + 1 = 3$
14. $\frac{a}{2} - 1 = 1$
15. $5 + 3c = 17$
16. $5n - 2 = n + 6$

17. In the statement $2a + 3a = 5a$, does a mean "any number" or "some number"?

18. In the statement $2a + 3 = 5$, does a mean "any number" or "some number"?

Using a *to represent one number and* a *and* b *to represent two numbers, write the following (Exs. 19–32) in algebraic symbols:*

19. Five more than a number is equal to twelve.
20. Five times a number is equal to thirty.
21. Two less than a number equals twenty-four.
22. Five less than three times a number equals sixteen.
23. Three more than twice a number equals fifteen.
24. A number increased by ten is equal to eighteen.
25. A number decreased by five is four.

[1] TO THE TEACHER. See Note 4 on page 458.

26. Four times a number decreased by seven is twenty-five.

27. Five times a number increased by three is twenty-seven.

28. The sum of two numbers is ten.

29. The difference between two numbers is two.

30. The product of two numbers is thirty-six.

31. One number divided by another is four.

32. One number increased by twice another is forty-two.

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33. How do the equations written in Exs. 28–32 differ from those in Exs. 19–27? Do the letters in these five equations represent "any number" or "some number"?

34. In the equation $a + b = 10$, there are two unknowns. a represents *any number*. After you have chosen a value for a, can b then be *any number?*

35. I have n two-cent stamps, twice as many three-cent stamps, and 7 five-cent stamps. State the number of three-cent stamps I have, using the letter n in your answer. What is the value of the n two-cent stamps? What is the value of all the three-cent stamps? State that the value of all the stamps together is $1.23.

Expressing One Quantity in Terms of Another

If one number is represented by n (or some other letter), and we know that a second number is 5 times the first, then the second number can be expressed as $5n$. If the second number is 5 more or 6 less than the first, we can represent it by $n + 5$ or $n - 6$. In such cases the second number is expressed *in terms of n.*

In solving problems and equations, it is often of advantage to express one quantity in terms of another because then it is necessary to find the value of only one of the unknowns.

(1) If b represents the number of planes built in one month, what will represent 12 times that many? ($12b$.)

(2) A girl is 4 years older than her brother. If the boy is x years old, how old is the girl in terms of x? ($x + 4$ years.)

Exercises

1. If n represents a certain amount of money, what will represent twice as much? What will represent 3 times as much?

2. If one side of a square is a inches, what is the perimeter in terms of a?

3. Represent one number by n. Express, in terms of n, another number which is 7 less than 3 times the first number.

4. If a certain amount of money is represented by n, what will represent an amount $2000 more than 4 times as much?

5. Let n represent the second number mentioned in each exercise below and then express the first number in terms of n.

(*a*) One number is five more than another.
(*b*) One number is five times another.
(*c*) One number is five less than another.
(*d*) One number is three more than twice another.
(*e*) One number is three less than twice another.
(*f*) One number is four less than three times another.

6. The length of a rectangle is seven times its width. If the width is w, what is the length in terms of w? What is the perimeter in terms of w?

7. The length of a rectangle is seven inches more than its width. If the width is w inches, what is the length in terms of w? What is the perimeter in terms of w?

8. The width of a rectangle is one third its length. If the length is l inches, what is the width in terms of l? What is the perimeter in terms of l?

9. The width of a rectangle is four inches less than the length. If the length is l feet, what is the width in terms of l?

10. A boy is seven years younger than his sister. If his sister is b years old, how old is the boy in terms of b? How old will each be in x years, in terms of b and x?

11. A mother is twice as old as her daughter. If the daughter is 20 years old, how old is the mother? How old was each x years ago in terms of x?

12. A watch costs seven dollars more than three times as much as a chain costs. If the chain costs n dollars, how much does the watch cost in terms of n? Make an equation stating that the cost of both together is forty-seven dollars.

13. An airplane travels thirty miles an hour less than twice as fast as another. Which of these rates would you represent by n? What is the other rate in terms of n?

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14. If one edge of a cube is n inches, what is the total length of all the edges in terms of n? If the total length is forty-eight inches, what is the value of n?

15. Florence has an allowance of $1.00 a week and saves one fourth of it. How many dollars will she save in n weeks?

16. Frank buys a pint of milk each day at school. Counting five days to the week, how many pints will he buy in a weeks and b days? How many quarts? How many gallons?

17. If on Monday your grade in algebra is m, on Tuesday n, on Wednesday p, on Thursday q, and on Friday r, what is your average for the five-day period?

A Study in Generalized Arithmetic †

You know how to find the cost of 8 pencils at 5 cents each. Do you know how to find the cost of a pencils at b cents each? This is a *general* problem, while the first one is a *specific* problem. The following will help you understand general problems.

(1) How would you find the total cost of any number of pencils of the same kind if you are told the cost of one pencil? (Multiply the cost of one pencil by the number of pencils.)

(2) Since there are *any number* of pencils and the cost of one pencil can be *any number* of cents, Ex. 1 may be stated algebraically in this way: What is the cost of a pencils at b cents each? What is the answer? (ab cents.)

Note that this answer, ab, has exactly the same meaning as the answer to Ex. (1), but that it is stated much more briefly when we use algebraic symbols.

(3) Restate the following question, using letters as in Ex. (2): How far will a man go if he travels a given number of hours at a given number of miles per hour? (How far will a man go in a hours at b miles per hour?) What is the answer? (ab miles.)

(4) If a oranges cost b cents, how much will one orange cost?

Be careful here. First ask yourself: If 3 oranges cost 18 cents, how much will one orange cost? The answer is 6 cents, obtained by *dividing* the 18 by 3. The answer to Ex. (4) is therefore $\frac{b}{a}$, indicating the division of the b by the a.

In working general problems, it is wise to substitute simple numbers for the letters, see what you would do to the numbers to get the answer, and then express the process by means of the letters.

(5) How many days are there in a weeks and b days?

Here again ask yourself a question like: How many days are there in 5 weeks and 3 days? What is the answer? (38 days.) How did you get it? (Multiplied the 5 by 7 and added 3.) Now go back to the original question, multiply the a by 7 and add the b. (Answer: $7a + b$.)

(6) A man gave a total of a dollars to his two children. If one received b dollars, how much did the other receive?

While arithmetic can be used to solve particular problems where numbers have definite values, with algebra we can solve general problems no matter what the values of the numbers may be. You have just had a few illustrations. In doing the following exercises you will have some easy practice in the use of the literal numbers of algebra in this general way.

Exercises

Before you write the answers to the following general problems, be sure to substitute specific numbers for the letters in order to discover the correct process to use.

1. What is the cost of 8 pencils at c cents each?

2. A boy buys m marbles and loses n marbles. How many has he left?

3. The difference between two numbers is 3. The larger one is a. What is the smaller number?

4. The difference between two numbers is d. The smaller one is t. What is the larger one?

5. If a plane flies for a hours at b miles an hour, how many miles will it go?

6. Two friends have x dollars between them. If one of them has y dollars, how much has the other?

7. Frank has five times as much money as James has. If James has n dollars, how much has Frank? How much do they have together?

8. The number of inches in the length of a rectangle is eight times the number of inches in the width. If the width is w inches, how long is the length? What is the perimeter?

9. How many months are there in a years and b months?

Chapter Summary

In this chapter you have learned that letters in algebra should not be thought of as letters but as numbers. Sometimes a letter means *any number;* at other times it means *some number* or *some numbers*. You have not only learned what the letters mean but you have had experience in using them. You have used them in expressing general arithmetic truths like $a \times 0 = 0$; $a + b = b + a$; if $a - b = c$, then $c + b = a$; and so on. You have used them in expressing one quantity in terms of another. And you have seen them used as unknowns in equations and have learned how to check the equations after finding, by trial, the value of the unknowns.

Besides these things you have learned how multiplication and division are expressed in algebra. You have found the value of expressions by substituting given numbers. You have combined like terms and solved some general arithmetic problems.

The following technical words have been explained:

Literal number	Unlike terms	Left member of equation
Arithmetic number	Combining terms	
Terms	Numerical coefficient	Right member of equation
Like terms	Equation	

Chapter Review[1]

1. Give one numerical example of each of the following general statements:
$a \times 0 = 0$, $a + b = b + a$, $ab = ba$, if $a - b = c$ then $c + b = a$, if $\frac{a}{b} = c$ then $bc = a$.

2. In the equation $4a - a = 3a$, does a represent *any number* or *some number?*

3. In the equation $4a - a = 9$, does a represent *any number* or *some number?*

4. In the expression $4 + a$, the a is a _?_ number and the 4 is an _?_ number.

5. The expression ab means a _?_ b.

6. What is the value of $5x$ when x is 7?

7. What is the value of $3a - 5$ when $a = 6$?

8. What does $3 \cdot 5$ mean? What does $(3)(5)$ mean?

9. In the expression $4a$, the 4 is called the _?_ _?_ of a.

10. When $2a + 3a$ is written as $5a$, the process of getting this result is called _?_ _?_.

11. How many terms has the expression $4a + 2b$?

12. In the expression $2a + 3b + 2b + 5a$, the like terms are _?_.

13. Unlike terms cannot be _?_.

14. The expression $\frac{a}{b}$ means a _?_ _?_ b.

15. To find the value of $\frac{a+b}{c}$ when $a = 2$, $b = 3$, and $c = 6$, I would first _?_ 2 and 3 and then _?_ the result by _?_.

16. Do $\frac{2}{3}b$ and $\frac{2b}{3}$ have the same values for all values of b?

17. $2n + 7 = 5$ is called an _?_. The left side is _?_, and the right side is _?_.

[1] To the Teacher. See Note 5 on page 459.

18. Like terms are combined by adding or subtracting the ? ?.

19. How many inches are there in f feet?

20. What is a simpler way of writing $3 \times a$?

21. What is a simpler way of writing $b + b + b + b$?

22. If a apples cost b cents, how much will 1 apple cost?

23. What is the value of the unknown in each of the following equations? Check your answer.

(a) $n + 2 = 5$ (c) $\frac{n}{2} = 6$ (e) $2n + 5 = 7$

(b) $n - 3 = 7$ (d) $5n = 30$ (f) $3n - 1 = 6$

24. Frank has a two-cent stamps and Martha has four less than twice as many. How many has Martha in terms of a?

25. Using n to represent *a number*, write the following as equations:

(a) A number divided by 3 is 6.
(b) Four more than a number is 10.
(c) Five less than a number is 2.
(d) Three times a number is 12.
(e) Five less than two times a number is 7.

26. Simplify by combining terms: $3a + 6b - 2a + 2b + 6$.

27. Find the value of the following when $a = 5$, $b = 3$, and $c = 1$:

(a) $3a + 5$ (b) $6b - 3$ (c) $\frac{a}{2} + bc$

28. A salesman has to travel m miles to meet a customer. How long will it take him if he averages r miles an hour?

29. A wholesale house sold a quantity of shoes at $2.40 a pair although they were originally priced at $4.00 a pair. At this reduction how much would a retailer save in buying n pairs?

30. Mr. Jones had b dozen eggs to sell. After he had sold n dozen, what fractional part of the b dozen had he sold? What fractional part did he have left?

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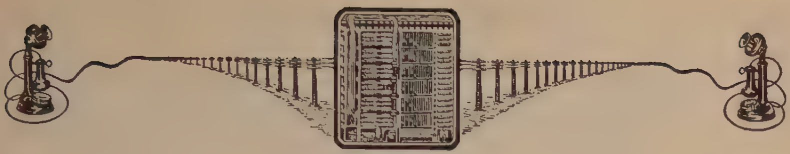
She studies figures from wind tunnel tests on Flying Fortresses. She is an "aerodynamicist," an expert on what makes planes fly. It takes 100,000 hours of mathematics to design a heavy bomber, and mathematics is constantly employed in improving airplanes and in flying them.



Chapter II

WHAT WE CAN DO WITH ALGEBRA[1]

> "Mathematicians, using the same tools — pencil and paper — used by high-school students in solving algebra and geometry problems, are busy helping engineers to discover new oil fields, to make telephones carry more conversations and 'talk better,' and to manufacture more efficient machinery of many kinds. There are many situations where the mathematician working hand in hand with the engineer can design a better machine, or plan how to build it quicker and cheaper, than can the engineer working alone."
>
> *Newspaper Statement*

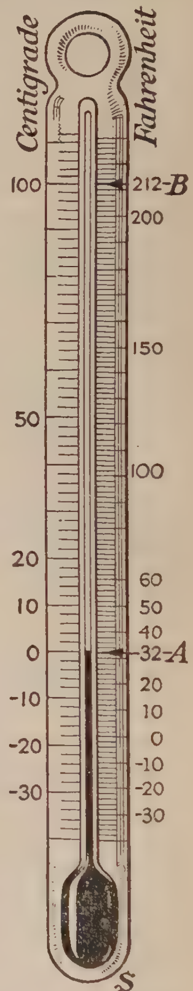
Wonderful things can be done with algebra, but not by the beginner. We cannot show you how to design an airplane, make a radio, or improve a piece of complicated machinery with the aid of mathematics. Years of experience are required for such work; you would need to know more algebra than you now do to understand even some beginning steps. We can, however, show you how to do some things by algebra that would be difficult or impossible for you to do using arithmetic alone.

In this chapter are samples of some ways in which algebra is used that will give you an idea of its problem-solving power. The four topics we have chosen for consideration are: (1) the direct use of formulas, by which answers are found by substituting values for the letters in the formula and then carrying through the indicated arithmetical processes; (2) solution of equations to find the values of unknown numbers; (3) the indirect use of formulas, where it is necessary to solve an equation after numbers have been substituted for the letters in the formula; (4) the solution of verbal problems.

[1] To the Teacher. See Note 6 on page 459.

Derivation and Meaning of a Formula

The Encyclopaedia Britannica says, "In health under normal conditions the temperature of man varies between 36° and 38° C."



From reading the encyclopedia, would you know and understand what this temperature is? The letter C is, of course, an abbreviation for "centigrade." But most persons in this country are not familar with the centigrade thermometer for measuring temperature. They do not know how hot or cold 36° C. really is. They ordinarily use the Fahrenheit thermometer (abbreviation, F.). The centigrade thermometer is used in science and in a number of foreign countries. The ordinary American needs to change these centigrade readings to Fahrenheit in order to understand them.

Algebra gives us a formula by which the change can be quickly made. But before using the formula, you should be familiar with the two scales for measuring temperature. Study of the scales explains the formula. Use the picture at the right to help you to complete the following statements:

(1) 32° F. corresponds to __?__ ° C.

(2) 100° C. corresponds to __?__ ° F.

(3) The letter A in the picture indicates the freezing point of water. This is __?__ degrees on the centigrade thermometer and __?__ degrees on the Fahrenheit thermometer.

(4) B indicates the boiling point of water. This is __?__ degrees on the centigrade thermometer and __?__ degrees on the Fahrenheit thermometer.

(5) On the centigrade thermometer there are __?__ scale units between the freezing point and the boiling point.

(6) On the Fahrenheit thermometer there are __?__ scale units between the freezing point and the boiling point.

(7) 68° F., the correct schoolroom temperature, is _?_ C.

Note that degrees below zero are indicated by a minus sign. Thus − 10° means 10° below zero.

(8) − 10° C. and − 20° C. correspond to _?_ F. and _?_ F.

We see that on the Fahrenheit scale there are 180 degrees, or scale units, between the freezing point and the boiling point of water. On the centigrade scale there are only 100 degrees between these points. Therefore the number of Fahrenheit degrees is $\frac{180}{100}$ or $\frac{9}{5}$ of the number of centigrade degrees. Putting it the other way round, there are $\frac{5}{9}$ as many centigrade degrees for a given change in temperature as there are Fahrenheit degrees.

From these facts of relationship between centigrade and Fahrenheit the formula for changing from one to the other is derived. The formula is the brief algebraic way of expressing these facts. The formula for changing from the centigrade to the Fahrenheit scale is —

$$F = \frac{9}{5}C + 32$$

Do not confuse the two uses of the letters F and C. In such statements as 68° F. and 20° C., the F and the C are abbreviations for the words "Fahrenheit" and "centigrade"; but in the formula, F and C are *numbers*. F is a number of degrees Fahrenheit and C is a number of degrees centigrade.

A ***formula*** is a statement in algebraic symbols expressing the relationship between two or more quantities.

Sometimes the formula is defined as an algebraic rule for computation.

Using the Formula

The problem which the encyclopedia gave us is to change 36° C. and 38° C. to the Fahrenheit scale. Let us now see how the formula can be used for making the change by substituting numerical values for the letters in it. The formula is —

$$F = \frac{9}{5}C + 32$$

(1) Change 36° C. to the Fahrenheit scale.

SOLUTION.

$$F = \frac{9}{5}C + 32$$
$$F = \frac{9}{5}(36) + 32$$
$$F = 64.8 + 32 = 96.8$$

36° C. is 96.8° F.

Note the form of the solution. First the formula is written. Then the given number is substituted for the letter. Finally the computation is carried out. Good form is an aid to efficiency and accuracy.

(2) Change 38° C. to the Fahrenheit scale. (This is left to you.)

If we wish to reverse the process and change a Fahrenheit reading to the centigrade scale, we can use another formula. The formula for changing from Fahrenheit to centigrade is —

$$C = \frac{5}{9}(F - 32)$$

Notice the use of the *parenthesis*. This is an algebraic symbol. It means that 32 is to be subtracted from F *first*.

Do what is in the parenthesis first.

(3) A good temperature for a schoolroom is 68° F. In a French schoolroom this temperature would be given in centigrade. What would it be?

SOLUTION.

$$C = \frac{5}{9}(F - 32)$$
$$C = \frac{5}{9}(68 - 32)$$
$$C = \frac{5}{9}(36) = 20$$

68° F. = 20° C.

(4) The melting point of iron is 2795° F. What is this in centigrade?

At this stage of your study of algebra it is necessary for us to use the two formulas for changing temperature readings, one for changing from Fahrenheit to centigrade and the other for reversing the process. A little later you will be able to derive either one of these formulas from the other. They are based upon the same facts; the relationships of the numbers are the same for both formulas. The problem is much the same as if I found out that you had twice as much money as I have. If I knew that, I should not have to be told that I have half as much as you have.

Exercises

1. Change the following centigrade readings to Fahrenheit. When the answer is fractional, give it to the nearest tenth of a degree. Be careful of your form of solution.

(*a*) 30° C.	(*d*) 125° C.	(*g*) 1250° C.
(*b*) 35° C.	(*e*) 62° C.	(*h*) 892° C.
(*c*) 100° C.	(*f*) 98° C.	(*i*) 453° C.

2. Change the following Fahrenheit readings to centigrade. Give fractional answers to the nearest tenth (Remember: In computing, do what is in the parenthesis first.)

(*a*) 41° F.	(*d*) 86° F.	(*g*) 1981° F.
(*b*) 50° F.	(*e*) 100° F.	(*h*) 9032° F.
(*c*) 59° F.	(*f*) 212° F.	(*i*) 148.5° F.

3. On a Fahrenheit thermometer such as physicians and nurses use in this country, normal body temperature is marked at 98.6°. What is this in centigrade?

4. A science book gives the melting point of aluminum as 660° C. What would this be in Fahrenheit?

5. In a certain textbook the melting point of copper is given as 1981° F. In an encyclopedia it is given as 1085° C. Are these the same temperature?

---

6. On a certain day the temperature dropped from 15° F. to − 10° F. How many degrees did it drop?

7. How many degrees would the temperature have to rise from − 13° F. to − 3° F.? from − 3° F. to 8° F.?

8. How much of a drop is it from − 3° F. to − 11° F.?

9. If the temperature drops 10° from 8° F., what will the reading then be?

10. If the temperature drops 8° from − 3° C., what will the reading then be?

11. What will the reading be if the temperature rises 10° from − 12° C.? from − 2° C.?

Order of Operations

When a series of operations is indicated in an algebraic expression, the order in which the operations are performed may be important. The following will make this fact clear and will indicate the order in which the operations are to be carried out.

If you were to find the value of $a + 2b$ when a is 3 and b is 5, you would substitute the numbers and get $3 + 2 \times 5$. You would see from the given expression $a + 2b$ that you should multiply first and get $3 + 10$, or 13. Suppose, however, you were given the example $3 + 2 \times 5$ without the original algebraic expression. Which operation, the addition or the multiplication, should you do first? You might add first and get 5×5, or 25. This would be wrong. In order to avoid confusion in evaluating (finding the value of) such expressions, mathematicians have agreed to the following rule:

> **In a series of operations, the multiplications and divisions should be performed first in the order in which they come and then the additions and subtractions.**

Evaluate the following:

(1) $3 \times 2 + 4$	(4) $5 + 6 \times 3$	(7) $9 + 3 \times 2 + 4$
(2) $3 + 2 \times 4$	(5) $8 + 2 \times 7$	(8) $10 - 4 \div 2 + 1$
(3) $9 \times 2 - 8$	(6) $12 - 3 \times 4$	(9) $12 + 6 \div 3 - 5$

As you have seen, a parenthesis may be used to indicate that the operations within it should be performed before the other operations are carried out. (See page 30.) Parentheses are often necessary as an aid in giving the order of operations.

Suppose, for example, you wish to write in symbols, "Multiply the sum of 8 and 5 by 2." You could not write $8 + 5 \times 2$, because according to the rule for "order of operations" you multiply first, and this would mean $8 + 10$. You could not write $2 \times 8 + 5$, for that would mean $16 + 5$. The use of a parenthesis takes care of the difficulty. The expression $2(8 + 5)$, in which you add the 8 and 5 first, means 2 times the sum of 8 and 5. What is the value of this expression?

> **In evaluating expressions involving parentheses, perform first the operations indicated within the parentheses.**

Study the following examples:

$$20 - (8 + 3) = 20 - 11 = 9$$
$$5(8 + 4) = 5(12) = 60$$
$$(2 + 3)(4 - 1) = (5)(3) = 15$$
$$2 + 3(4 - 1) = 2 + 3(3) = 2 + 9 = 11$$

Note that in the last example, the multiplication is done before the addition.

Exercises

Evaluate the following expressions:

1. $5 \times 3 + 4$
2. $3 + 4 \times 5$
3. $8 \times 3 - 4$
4. $10 - 5 \times 2$
5. $7 + 3 \times 5$
6. $8 - 4 \div 2$
7. $8 \div 2 + 4$
8. $6 - 3 \times 2 + 8$
9. $2(8 + 3)$
10. $5(9 - 6)$
11. $4 + (8 - 5)$
12. $9 - (2 + 3)$
13. $10 - (6 - 2)$
14. $(5 + 2)(8 + 1)$
15. $(9 + 3)(6 + 4)$
16. $9 + 3(6 + 4)$
17. $(9 - 3)(6 + 4)$
18. $100 - 3(6 + 4)$
19. $5(10 - 3)$
20. $64 - (10 + 12)$
21. $5 + (9 - 2)$
22. $7(6 + 3 - 5)$
23. $2(3 + 4)(7 - 5)$
24. $20 + 2(7 - 3) - (8 - 2)$

25. Find the value of each of the following expressions, using the values given for the letters in each exercise.

(*a*) $a(b + c)$; $a = 5, b = 2, c = 4$

(*b*) $a + (b + c)$; $a = 3, b = 1, c = 2$

(*c*) $a - (b + c)$; $a = 10, b = 3, c = 5$

(*d*) $(a + b)(c - d)$; $a = 10, b = 2, c = 5, d = 1$

---

26. If you wish to subtract the quantity $b - c$ from a, how would you express this fact?

27. The area of a rectangle is found by multiplying the length by the width. If the length of a rectangle is $a + b$ feet and the width is c feet, how could you express the area?

28. Express the area of a rectangle whose length is $a + b$ and whose width is $a - b$.

29. An automobile was driven at the rate of $r + 5$ miles an hour for 3 hours. Express the distance traveled.

30. How would you write, using an equation, that $n + 6$ is 3 times the quantity $n - 1$?

Solving Equations †

In the preceding chapter you learned that the letter in an equation means "some number" which is called the unknown. You found the value of the unknown number *by trial* and then checked your result. You will now learn some definite algebraic methods of finding the value of the unknown in an equation. (Always try to answer the questions before looking at the answers that follow them.)[1]

(1) I am thinking of a number. If I add 2 to it, the result is 6. What is the number? (The answer is 4.)

How did you get the number? (I subtracted 2 from the 6.)

By using an equation, the problem in Ex. (1) can be stated briefly thus: If $n + 2 = 6$, what is n? The answer is obtained as above by subtracting 2 from 6.

(2) If $n + 5 = 11$, what is n? (6.) How did you get the 6? (By subtracting 5 from 11.)

(3) I am thinking of a number. If I subtract 2 from it, the result is 6. What is the number? (8.)

How did you get the number? (I added 2 to the result.)

Using an equation, Ex. (3) can be stated briefly: If $n - 2 = 6$, what is n? The answer is obtained as in Ex. (4) by adding 2 to 6.

[1] TO THE TEACHER. See Note 7 on page 459.

(4) If $n - 5 = 11$, what is n? (16.) How did you get the 16? (By adding 5 to 11.)

(5) I am thinking of a number. If I multiply it by 2, the result is 6. What is the number? (3.)

How did you get the number? (I divided the result by 2.)

Again using an equation, Ex. (5) can be stated briefly: If $2n = 6$, what is n? The answer is obtained as in Ex. (5), by dividing the 6 by 2.

(6) If $5n = 10$, what is n? (2.) How did you get the 2? (By dividing the 10 by 5.)

(7) I am thinking of a number. If I divide it by 2, the result is 6. What is the number? (12.)

How did you get the number? (I multiplied the result by 2.)

In equation form Ex. (7) can be stated briefly: If $\frac{n}{2} = 6$, what is n? The answer is obtained as in Ex. (7), by multiplying the 6 by 2.

(8) If $\frac{n}{3} = 7$, what is n? (21.) How did you get the 21? (By multiplying the 7 by 3.)

If you examine carefully the exercises which you have just done, you will note the following important points: When in Ex. (1) I said that I had *added* 2 to a number, you then *subtracted* 2 from the result to "get back" to the number. When in Ex. (3) I *subtracted* 2 from a number, you added 2 to the result to "get back" to the number. When I *multiplied* a certain number by 2, you *divided* the result by 2 to "get back" to the number. When I *divided* a number by 2, you *multiplied* the result by 2 to "get back" to the number.

(9) Find the unknown numbei in each of the following equations. Check your answers.

$$n + 3 = 12 \qquad \frac{n}{3} = 12$$

$$n - 3 = 12 \qquad 3n = 12$$

For each equation you wish to solve you should ask yourself, "What has been done to the unknown?" When you have answered this question, *do the opposite* thing to "get back" to the unknown. For example, in the equation $n + 3 = 12$, 3 has been *added* to n; so you must *subtract* 3 from 12 to "get back" to n.

Study the following table, which shows for each of the four processes what is the opposite or ***inverse process.***

Process	Opposite or Inverse Process
Addition	Subtraction
Subtraction	Addition
Multiplication	Division
Division	Multiplication

(10) Find the value of the unknown in each of the following equations. Check your answers.

$$n + 15 = 31 \qquad 13\,n = 117$$

$$\frac{n}{12} = 36 \qquad n - 42 = 67$$

To find the value of the unknown in such simple equations as you have been studying, first determine what has been done to the unknown to get the result. Then carry out the inverse process on the result to find the value of the unknown.

(11) Discuss several equations selected at random from the list of exercises on page 38. Tell what has been done to the unknown in each case and what you will do to find the value of the unknown.

Finding the value of the unknown in an equation is called ***solving the equation.*** To solve an equation is to find that value of the number which, when substituted for the unknown in the equation, will make the left side equal to the right side. This value of the unknown is called the ***root*** of the equation. The root is said to ***satisfy*** the equation.

From Loggon's "Oxonia Illustrata," courtesy of Avery Library, Columbia University

Oxford University, England, where Robert Recorde, the inventor of the equality sign, taught four hundred years ago.

Invention of Equality Sign

Almost from the very beginning of your study of arithmetic you have used the equality sign (=). You have not known mathematics without it, and as you begin work with equations we assume that you are familiar with it.

But there was not always an equality sign for mathematicians to use. The symbols and the operations of mathematics have gradually been discovered or invented during a long period of time. Although the first known record of the use of the equation goes back more than three thousand years, to around 1700 B.C., there was no equality sign to work with during most of that time. The sign is a recent invention compared with the life of algebra and the equation.

Formerly, instead of the equality sign the word "equals" was written in the equation. The sign was invented by Robert Recorde of Wales, who lived from 1510 to 1558 and taught mathematics in Oxford University. He explained his reason for introducing the sign thus:

> "And to avoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke vse, a paire of paralleles, or Gemowe [twin] lines of one lengthe, thus: ═══, bicause noe .2. thynges can be moare equalle."

Exercises

Solve the following equations. Check your answers. Use the form of the sample.

SAMPLE. $n + 7 = 12$

$n = 5$

CHECK. $5 + 7 = 12$

$12 = 12$

1. $4n = 20$
2. $n + 4 = 20$
3. $n - 4 = 20$
4. $\frac{n}{4} = 20$
5. $a + 3 = 9$
6. $\frac{a}{3} = 9$
7. $a - 3 = 9$
8. $3a = 9$
9. $x - 7 = 6$
10. $b + 4 = 12$
11. $3p = 12$
12. $5n = 30$
13. $8p = 24$
14. $\frac{x}{2} = 1$
15. $\frac{1}{2}x = 5$
16. $\frac{1}{5}x = 5$
17. $6x = 6$
18. $6n = 0$
19. $c + 4 = 5$
20. $2n = 12$
21. $3n = 18$
22. $5n = 25$
23. $\frac{n}{3} = 2$
24. $\frac{a}{5} = 2$
25. $\frac{b}{3} = 5$
26. $p - 5 = 8$
27. $b - 3 = 9$
28. $\frac{1}{4}d = 3$
29. $n + 3 = 9$
30. $n - 3 = 9$
31. $3n = 9$
32. $\frac{n}{3} = 9$
33. $4a = 16$
34. $a - 4 = 16$
35. $a + 4 = 16$
36. $\frac{a}{4} = 16$
37. $n - 1 = 2$
38. $6a = 18$
39. $n - 3 = 5$
40. $7n = 14$
41. $n + 2 = 3$
42. $n + 6 = 6$
43. $a - 6 = 0$
44. $p - 8 = 0$
45. $r - 10 = 0$
46. $a + 5 = 18$
47. $\frac{n}{5} = 6$
48. $\frac{x}{5} = 10$
49. $\frac{c}{4} = 20$
50. $\frac{x}{3} = 5$
51. $\frac{1}{6}x = 6$
52. What value of n satisfies the equation $n + 6 = 13$?
53. What is the root of the equation $2n = 36$?
54. What value of n makes the left side of the equation $n - 5 = 17$ equal to the right side?

Solve the following equations and check your answers:

55. $n + 2.4 = 9.6$

56. $d - 3.5 = 9.2$

57. $p + 3.7 = 6.2$

58. $k - 4.7 = 3.5$

59. $n - \frac{1}{2} = 4$

60. $p + 2\frac{1}{3} = 5$

61. $n - 3\frac{2}{3} = 5\frac{1}{2}$

62. $p + 2\frac{1}{2} = 6\frac{1}{4}$

63. $\frac{a}{21} = 63$

64. $\frac{r}{14} = 140$

Solve the following equations for n:

65. $n - a = b$

66. $n + a = b$

67. $an = b$

68. $\frac{n}{a} = b$

Solving Another Kind of Equation†

Can you solve the equation $2n + 5 = 17$ without trying several numbers? If you can, tell how you did it. If you cannot do it, the following discussion will show you how it is done.

(1) If 5 is added to twice a number and the result is 11, what is twice the number? (6.) How did you get the 6? (I subtracted 5 from the 11.)

(2) If, then, twice the number is 6, what is the number? (3.)

Exs. (1) and (2) may be stated briefly: "If $2n + 5 = 11$, what is the value of $2n$?" and "If $2n = 6$, what is n?" You found the value of n in two steps instead of in one step, as you have done with preceding equations.

(3) In the equation $2n + 5 = 11$, how would you know that you should *subtract* 5 to find the value of $2n$? (I see that 5 has been *added* to $2n$.)

This equation should be solved in two steps as follows:

$$2n + 5 = 11$$
$$2n = 6$$
$$n = 3$$

CHECK.
$$2 \times 3 + 5 = 11$$
$$6 + 5 = 11 \text{ and } 11 = 11$$

(4) You should be able to solve the equation $2n - 5 = 11$ without help. Solve it and check your answer.

SOLUTION.
$$2n - 5 = 11$$
$$2n = 16$$
$$n = 8$$

CHECK.
$$2 \times 8 - 5 = 11$$
$$16 - 5 = 11 \text{ and } 11 = 11$$

(5) Solve and check the following equations:

(a) $5n + 3 = 23$
(b) $5n - 3 = 7$
(c) $2n + 5 = 7$
(d) $2n - 5 = 7$
(e) $3n - 1 = 8$
(f) $3n + 1 = 13$

Exercises

Solve and check the following equations:

1. $5n + 3 = 33$
2. $5n - 3 = 17$
3. $2n + 3 = 6$
4. $2n + 5 = 17$
5. $3n - 4 = 17$
6. $2a + 3 = 15$
7. $6a + 1 = 37$
8. $8a - 7 = 41$
9. $2b + 3 = 21$
10. $3a + 5 = 32$
11. $9a - 1 = 80$
12. $4a + 17 = 53$
13. $3b - 5 = 12$
14. $6x + 17 = 37$
15. $9a - 8 = 73$
16. $8n + 22 = 70$
17. $6n - 36 = 0$
18. $5a - 40 = 0$
19. $5n + 6 = 41$
20. $18n - 7 = 29$
21. $n + 4 = 12$
22. $n - 4 = 12$
23. $4n = 12$
24. $\frac{n}{4} = 12$
25. $9n = 981$
26. $n + 32 = 96$
27. $n - 41 = 52$
28. $\frac{n}{32} = 8$
29. $4a - 7 = 13$
30. $5b + 2 = 17$
31. $9d - 4 = 14$
32. $3n + 10 = 28$
33. $2n + 7 = 12$
34. $3a - 5 = 16$
35. $6a - 1 = 20$

36. $3b + 5 = 9$

37. $x + 3 = 7$

38. $x - 2 = 3$

39. $\frac{n}{3} = 7$

40. $2n + 3n = 45$

41. What is the root of the equation $2a - 5 = 13$?

42. What value of n satisfies the equation $3n + 8 = 23$?

43. What value of n makes the left side of the equation $5n - 24 = 6$ equal to the right side?

Solve and check the following equations:

44. $3n = 2$ (Ans. $n = \frac{2}{3}$)

45. $4n = 3$

46. $7n = 5$

47. $2n = 3$

48. $3n = 4$

49. $5n = 7$

50. $4a = 9$

51. $9a = 4$

52. $3 = 2n$

53. $2 = 3n$

54. $\frac{x}{2} + 1 = 5$

55. $\frac{x}{3} - 1 = 7$

56. $2.5a = 6.25$

57. $\frac{x}{0.6} = 8$

58. $1.8b = 36$

59. $c - .07 = .83$

60. $0.5m = 3$

61. $\frac{1}{2}x = \frac{3}{4}$

62. $\frac{1}{2}x + \frac{3}{4} = 1\frac{1}{2}$

63. $0.6a = .12$

64. $3.2y = 80$

65. $3n + 5 = 7$

66. $5x - 3 = 18$

67. $2n + 1.8 = 2.5$

68. $3n + 1.2 = 7.2$

69. $3n - 1.2 = 7.2$

70. $\frac{n}{4} - 2 = 10$

71. $\frac{a}{0.62} = 1.45$

72. $.64b = .8448$

73. $2.5r - 2.0 = 48$

74. $.06t + 43.5 = 87.66$

75. $\frac{x}{1.2} + 3.4 = 9.7$

76. $\frac{x}{2.3} - 4.5 = 8.6$

Combining Terms in Equations

The first step in solving equations that have like terms on either side is to combine the like terms. Review page 11 if necessary.

EXAMPLE. $5x + 6 - 2x + 3 = 27$

Combining terms, $3x + 9 = 27$

$$3x = 18$$

$$x = 6$$

CHECK. $30 + 6 - 12 + 3 = 27$

$$27 = 27$$

Exercises

Solve and check the following equations:

1. $5n + 2n = 21$
2. $7n - 3n = 20$
3. $4a + a = 25$
4. $b + 3b = 12$
5. $5a + a - a = 35$
6. $9a - 3a + 2a = 16$
7. $12n + n - 11n = 5$
8. $5n - 3 + 2n = 18$
9. $3x + 6 - 2x = 14$
10. $n + 2n + 3 + 4n = 24$
11. $6 + 2n - 4 = 5$
12. $4n + 3 - 2n = 7$
13. $n + 2n + 3 + 4n = 24$
14. $3b + 5 + 4b - 2 = 38$
15. $2x + 7 + 3x - 2 = 35$
16. $18y + 12 - 11y + 4 = 23$
17. $12x + 8 - 4 = 12$
18. $5n + 10 + 3 - 2n = 26$
19. $6x - 5x + 15 + 5 = 35$
20. $5n + 15 - 3n = 23$

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21. $\frac{1}{2}n + \frac{3}{4}n - n = 2.5$
22. $1.5x - .2x + 1.2x = .25$
23. $1.3n - .4n + .03n = 3.72$
24. $c + 5 + .05c = 11.3$

Indirect Use of the Formula

Do you remember the three "cases" of per cent problems? They are —

Case I. Finding a per cent of a number; for example, what is 68% of 432?

Case II. Finding what per cent one number is of another; for example, 18 is what per cent of 47?

Case III. Finding a number when a per cent of it is known; for example, 432 is 68% of what number?

Can you do these three examples now? Do Cases II and III confuse you? With the use of algebra the three cases of per cents are easy. You need not even consider what case the example is. In all three cases, you will substitute numbers in the same formula.

EXAMPLE, *Case I.* Find 6% of 234.

Here the number 234 is called the *base;* 6% is the *rate;* and the answer to be found is the *percentage.* As you know, the rule is: To find the percentage, multiply the base by the rate (expressed as a decimal). Stating this rule as a formula, we have

$$p = br \qquad \text{(Per cent formula)}$$

In the formula, p is the percentage, b the base, and r the rate expressed as a decimal. There are three numbers in this formula, p, b, and r. Before attempting the solution of a problem by the use of the formula you should ask, "Do I know p? Do I know b? Do I know r?"

To solve the example above, we now substitute in the formula the given values of b and r. $b = 234$, $r = .06$.

SOLUTION.

$$p = br$$
$$p = (234)(.06)$$
$$p = 14.04$$

234
.06
14.04

EXAMPLE, *Case II.* 30 is what per cent of 48?

We use the same formula, $p = br$. The percentage, p, is 30; the base, b, is 48; and the rate, r, is unknown.

SOLUTION. $p = br$

Substituting the numbers, $30 = 48r$

If it helps you, write this as $48r = 30$

You now have an easy equation to solve according to the definite rules which you have learned.

$$r = \frac{30}{48} = \frac{5}{8} = .625$$

$$\begin{array}{r} .625 \\ 8\,\overline{)\,5.000} \end{array}$$

CHECK.

$$\begin{array}{r} 48 \\ .62\tfrac{1}{2} \\ \hline 24 \\ 96 \\ 28\,8 \\ \hline 30.00 \end{array}$$

Note that the solution gives the rate as a decimal, which must be changed to per cent.

ANSWER. 30 is $62\frac{1}{2}\%$ of 48.

CHECK. $62\frac{1}{2}\%$ of 48 is 30.

EXAMPLE, *Case III.* 32 is 15% of what number?

Again we use the same formula, $p = br$. In this example we have $p = 32$, $r = .15$, and b is unknown.

$$\begin{array}{r} 2\,13\tfrac{1}{3} \\ {}_{\times}15.\,\overline{)\,32_{\times}00.} \\ 30 \\ \hline 20 \\ 15 \\ \hline 50 \\ 45 \\ \hline 5 \end{array}$$

SOLUTION. $p = br$

Substituting the numbers, $32 = b(.15)$

Write this as $.15b = 32$

Solve the equation, $b = \frac{32}{.15} = 213\frac{1}{3}$

ANSWER. 32 is 15% of $213\frac{1}{3}$

CHECK. 15% of $213\frac{1}{3}$ is 32

CHECK.

$$\begin{array}{r} 213\tfrac{1}{3} \\ .15 \\ \hline 5 \\ 1065 \\ 213 \\ \hline 32.00 \end{array}$$

Case I problems of percentage are easy by arithmetic. Cases II and III are likely to be confusing because these problems make use of the meaning of percentage *indirectly*. The use of the algebraic formula and methods of solving equations makes it possible to treat all three cases as one.

Exercises

1. Using the formula $p = br$, find r when p and b are as follows. (Give your answer as a per cent to the nearest tenth of a per cent.)

(a) $p = 18, b = 72$
(b) $p = 32, b = 160$
(c) $p = 15, b = 45$
(d) $p = 38, b = 64$
(e) $p = 64, b = 38$
(f) $p = 61, b = 34$

2. Using the formula $p = br$, find b when p and r are as follows:

(a) $p = 32, r = .05$
(b) $p = 21, r = .06$
(c) $p = 58.5, r = .13$
(d) $p = 37, r = .04$
(e) $p = 28.17, r = .035$
(f) $p = 55.44, r = 1.32$

3. Find 4% of 262.

4. 24 is what per cent of 262?

5. 38.4 is what per cent of 32?

6. What is $\frac{1}{2}$ of 1% of $62.50?

7. If the discount on an article is $8.32 and the rate of discount is 8% of the marked price, what is the marked price?

8. During a sale a merchant sold 183 shirts or 75% of his stock. How many shirts did he have in stock before the sale?

9. An electrical appliance store advertised a vacuum cleaner for $2.13 down. They stated that this was 2.5% of the price. What was the price?

10. In an algebra test, John had 24 out of 30 examples correct. What per cent of the 30 examples did he have correct?

11. Our school lost five of its twelve games in basketball last season. What per cent of its games did it win?

12. At a sale a hat which was originally marked to sell at $8 was sold for 10% less than the marked price. What was the amount of the discount? What was the selling price?

13. The amount of a bill is $246.75. A discount of 5% is given for cash. What is the net amount (the amount after subtracting the discount)?

14. A pair of shoes originally marked for sale at \$4 was sold for \$2.20. What was the rate of discount?

15. My class has raised \$325 and has reached 65% of its quota. How much more does it need to raise?

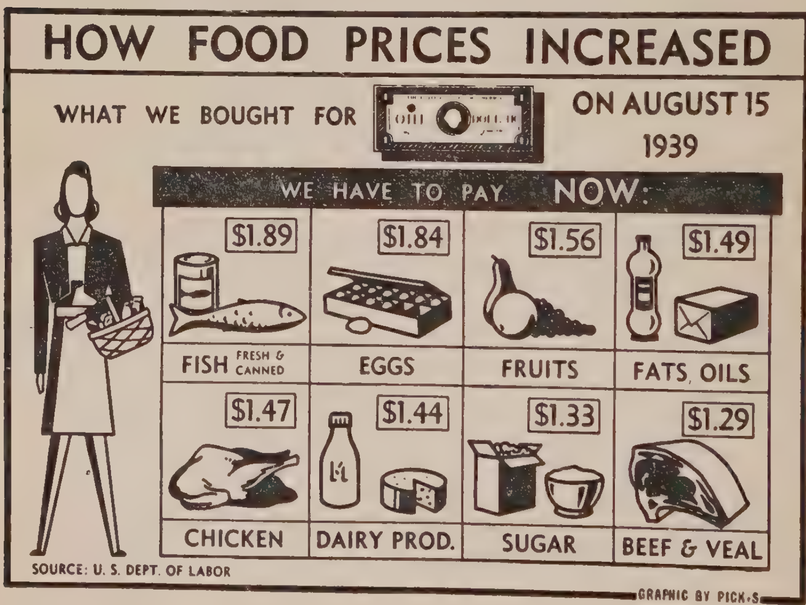
16. A man bought an article for \$6.16 and wished to sell it for an amount that would give him a profit of 23% on the selling price. For what price should he sell it?

17. Our city carried on an intensive safety campaign during one year. As a result the number of accidents was reduced from 480 to 264. What was the per cent of decrease?

18. Last year butter sold for 35 cents a pound. This year it is 52 cents a pound. What is the per cent of increase?

19. The chart below appeared in a newspaper early in 1943. It shows per cents of increase in food prices since the middle of August, 1939. Explain.

On the basis of this chart, what did a dozen eggs cost in 1943 if a dozen cost 35 cents in 1939? If a pound of beef sold for 52 cents (on the average) in 1943, what was the price in 1939?



Multiplying an Indicated Sum by a Number †

The *sum* of 2, 3, and 5 is 10. The expression $2 + 3 + 5$ is an *indicated sum.* In algebra we often work with indicated sums. The expression $x + y$ is an indicated sum; $a + 7 + c$ is an indicated sum.

(1) Suppose you need to find the value of 5 times the sum of 3 and 4. Can you multiply 5 by 3 and add 4 or must you multiply both the 3 and the 4 by 5 and then add? (The latter.)

(2) Does $6(4 + 5)$ equal $(6)(4) + 5$ or does it equal $(6)(4) + (6)(5)$? (The latter.)

(3) Using a, b, and c to represent any three numbers, write the general principle that is involved in Exs. (1) and (2). (The principle is $a(b + c) = ab + ac$.)

The statement $a(b + c) = ab + ac$ is called the ***distributive law*** of algebra.

(4) State the distributive law in words. (The result of multiplying the sum of b and c by a is the same as the result of multiplying a by b, then a by c, and then adding these two products.)

(5) Illustrate the distributive law by the use of several sets of numbers. (For example, if a is 2, b is 3, and c is 4, $2(3 + 4) = 2(7) = 14$. Also $2(3) + 2(4) = 6 + 8 = 14$.)

(6) This law is true no matter how many terms there are in the parenthesis. For example, $a(b + c + d + e) = ab + ac + ad + ae$. Check this by several numerical examples.

(7) Does $a(b - c) = ab - ac$? (Yes. For example $3(5 - 2) = 3(5) - 3(2)$. Until you have studied *negative numbers* you must consider b greater than c.)

(8) Does $2(n + 3) = 2n + 3$ or does it equal $2n + 6$? (It equals $2n + 6$. Check by means of any number for n.)

(9) $3(r + 5) =$ __?__. ($3r + 15$.)

(10) $r(s + t + u) =$ __?__. ($rs + rt + ru$.)

(11) $5(n + 6 + b) =$ __?__. ($5n + 30 + 5b$.)

(12) $6(n - 3) =$ __?__. ($6n - 18$.)

Another way to state the distributive law is this:

To multiply an indicated sum (or difference) by a number, multiply *every* term of the indicated sum (or difference) by the number.

(13) Multiply as indicated:

(*a*) $2(n + 6)$ (*c*) $a(r + s + t)$
(*b*) $3(n - a)$ (*d*) $m(r + 6)$

(The answers are (*a*) $2n + 12$, (*c*) $ar + as + at$,
(*b*) $3n - 3a$, (*d*) $mr + 6m$.)

Exercises[1]

Multiply as indicated:

1. $2(a + b)$
2. $3(a - b)$
3. $a(b + d)$
4. $2(n + 3)$
5. $5(n - 7)$
6. $3(r - s)$
7. $3(r + 5)$
8. $5(n - 1)$
9. $4(x + 7)$
10. $11(4 + x)$
11. $5(7 - a)$
12. $7(n + 5)$
13. $4(x - 6)$
14. $a(b - 3)$
15. $3(a + b + c)$
16. $5(a + b - c)$
17. $9(a - b + c)$
18. $4(a + 6 + c)$
19. $a(b + r - s)$
20. $7(ab + cd)$

Multiply as indicated and combine like terms:

21. $2(n + 5) + 3n$
22. $4(n - 6) - 2n$
23. $6(a + 2) - 5$
24. $6n + 5(2 - n)$
25. $4 + 5(b + 2)$
26. $8 + 3(r - 2)$
27. $ab + a(b + c)$
28. $4 + 3(n + 6) + 7n$

Solve and check the following equations:

29. $2(n + 2) = 9$
(SUGGESTION. $2n + 4 = 9$)
30. $3(n + 5) = 24$
31. $4(n + 3) = 17$
32. $2(n - 3) = 8$
33. $9(n - 7) = 12$
34. $4(n + 2) = 15$
35. $2(n + 6) = 17$
36. $3(n - 5) = 20$
37. $6(n + 2) - 12 = 18$

[1] TO THE TEACHER. See Note 8 on page 459.

Multiplying an Indicated Product by a Number †

As you know, the *product* of 2, 3, and 4 is 24. The expression $2 \times 3 \times 4$ is an *indicated product*. These expressions are indicated products: ab; $4m$; $2xy$.

Each of the numbers in an indicated product is a *factor* of the product. Thus the 2, 3, and 4 are factors of the expression $2 \times 3 \times 4$. The factors of $4m$ are 4 and m.

The rule for multiplying an indicated product by a number is in direct contrast to the rule for multiplying an indicated sum by a number. You multiply only a single factor of the product and not all the factors by the number.

(1) How much is $4(3 \times 2)$? (24.) Did you multiply by the 4 once, twice, or three times? (Just once.)

(2) Does $4(3 \times 2)$ equal $(4)(3) \times (4)(2)$? (No. You should multiply by 4 only once.)

(3) Note the following examples:

(*a*) $2 \times 3 \times 2 = 6 \times 2$ (*d*) $2 \times 3 \times 5 = 6 \times 5$
(*b*) $2 \times 3 \times 3 = 6 \times 3$ (*e*) $2 \times 3 \times a = 6a$
(*c*) $2 \times 3 \times 4 = 6 \times 4$ (*f*) $2 \times 3b = 6b$

Check the truth of the first four examples. Note that the fifth and sixth examples follow the same principle as the first four examples. The rule is stated below.

To multiply an indicated product by a number, multiply only *one* (any one) of the factors by that number.

(4) Show by performing the multiplications that $2(3 \times 5 \times 7) = 6 \times 5 \times 7$ or $3 \times 10 \times 7$ or $3 \times 5 \times 14$. How were these expressions obtained? (By multiplying each of the factors 3, 5, and 7 in turn by 2.)

(5) Study the following examples:

(*a*) $2(5)(7) = 10(7)$ (*d*) $3(4n + 5) = 12n + 15$
(*b*) $2(5b) = 10b$ (*e*) $3(4n - 5) = 12n - 15$
(*c*) $b \times 3c = 3bc$ (*f*) $a(2b + 7) = 2ab + 7a$

Exercises

Multiply as indicated:

1. $5 \times 3b$
2. $6(3b)$
3. $2(4a)$
4. $5(2cd)$
5. $a(bc)$
6. $7 \times 3r$
7. $a \times 3r$
8. $b(rs)$
9. $b(7t)$
10. $9(7cd)$
11. $4(n+5)$
12. $3(r-s)$
13. $a(t+u)$
14. $3(4n+3)$
15. $2(3a+b)$
16. $a(2c+3d)$
17. $9(7+2y)$
18. $6(3a-7)$
19. $5(4b-3)$
20. $3(3a-2b)$

Multiply as indicated and combine like terms:

21. $2(3a+2b)-4a$
22. $3(4a+5)+6$
23. $8+5(2b+7)$
24. $9b+3(3a+2b)$
25. $8+4(n-2)$
26. $24+3(7n-3)$
27. $8a(b+c)-3ab+2ac$
28. $7(3a+2b-5c)-10b$

Solve and check the following equations:

29. $3(2n+1)=27$
30. $2(3n+5)=27$
31. $3(4x-3)=33$
32. $2(3x-1)=14$
33. $4(n+7)=50$
34. $6(n+2)-12=18$
35. $3(n+5)=20$
36. $3(n+4)+10=43$
37. $4(3x+2)-4=12$
38. $5(n+3)-3n=23$

A Magician's Trick?

Write down your house number. Multiply it by 2. Add 10. Multiply the result by 50. Add the number of days in a year. Add your age. Subtract 865.

Look at the final answer. The two figures on the right should be your age and the rest of it your house number. Is this a magician's trick? Let algebra show you that it is not.

Let your house number be represented by a and your age by b. Then the first paragraph above can be stated as —

$$50(2a+10)+365+b-865$$

and this simplified is $100a+b$.

In other words, the whole thing simmers down to 100 times your house number plus your age. The rest is put in to distract your attention from the essential simplicity of the computation. Multiplying your house number by 100 moves it over two places so that when your age is added it stands out by itself as the last two figures.

This trick will not work if you are over 99 years old. Why?

Practice in Algebraic Expression

1. If n represents one number, what represents a number that is —

(*a*) 5 times as much?

(*b*) 5 more than n?

(*c*) 6 less than n?

(*d*) 7 more than twice as much?

(*e*) 4 less than three times as much?

(*f*) the sum of (*d*) and (*e*)?

2. If the difference between two numbers is 12 and the smaller number is n, what is the larger?

3. If the difference between two numbers is 12 and the larger number is n, what is the smaller?

4. If one part of 36 is n, what is the other part?

5. If the length of a rectangle is 8 in. and the width is 5 in., what is the perimeter?

6. If the length of a rectangle is 3 in. more than the width and the width is w, what is the length? What is the perimeter?

7. How far will a plane travel if it goes 300 miles an hour for 5 hours? n miles an hour for 5 hours? $2n$ miles an hour for 6 hours?

8. What is the total value of 17 2-cent stamps? of n 3-cent stamps? of $5n$ 3-cent stamps? of $a - 3$ 2-cent stamps?

9. State by means of an equation that —

(*a*) the sum of n, $2n + 5$, and $n - 2$ is 72.

(*b*) the difference between $4n$ and n is 36.

Analysis and Solution of Verbal Problems†

When you have a problem that is stated in words to solve, the first important thing to do is to make sure that you know in every detail the facts and relationships that are given. Let us take a sample verbal problem.[1]

PROBLEM. Of three numbers, the first number is 5 more than 3 times the third and the second is 2 less than the third. The sum of the first and second numbers is 59. What are the three numbers?

Careful reading shows at once that this would be a difficult problem by means of arithmetic. While ordinarily careful reading may help you understand the problem, even then it may still seem complicated. You need to "take the problem apart" — to analyze it — in order to be sure of the relationships among the numbers.

One of the best ways to become entirely familiar with a problem is to take some number at random as a possible solution and then check to see if you guessed right. For the purpose of understanding the problem, it is not essential that you guess right. In checking the guess you become so familiar with the relationships in the problem that the method of procedure in solving it by algebra becomes clear. This first step is important. If, after taking a number at random as a possible answer, you do not know how to check it, there is little likelihood that you can solve the problem by any method.

(1) Read the sample problem again. What do you think the numbers are? Make a guess. Here are five sets of answers given by one class.

First number	3	15	5	25	29
Second number . . .	9	20	30	90	20
Third number	12	30	40	100	18

(2) The problem gives several relationships among the numbers. Read the first relationship stated in the problem. (The first number is 5 more than 3 times the third.)

[1] TO THE TEACHER. See Note 9 on page 460.

(3) Check the first set of answers above by this relationship. Does it check? (No. 5 more than 3 times 12 is 41, not 3.) Check the other answers in the same way. Do you think the members of this class gave much attention to the relationships stated in the problem? They evidently did not know the details.

(4) Do you now know which one of the numbers you should guess first? Which one? (The third number.) Why? (Because the others depend upon it for their values.)

(5) Now try again to give your answer to the problem. This time write down the third number first; then, using the first given relationship, write down the first number. Look for the second relationship stated in the problem. What is it? (The second is 2 less than the third.) Now write down the second number.

(6) Below are five sets of answers given by the same class on their second trial. Go through these answers and check them against the first relationship and then check them against the second relationship. Which answers check against both relationships? (The second, fourth, and fifth.)

First number	20	17	47	20	83
Second number . . .	55	2	15	3	24
Third number	65	4	14	5	26

(7) One more relationship is stated in the problem. What is it? (The sum of the first and the second numbers is 59.) Do any of the above answers check against this condition? (No.)

The purpose of the discussion of this problem so far has been to help you become thoroughly familiar with the details of the problem — to find "what it is all about." The algebraic solution follows exactly what you have already done, as you will see.

(8) You know that the answer is "some number." How do you express "some number" in algebra? (By a letter, say n.) You found that it was wise to guess the third number first; so let the third number be represented by n.

(9) How did you get the first number after you had guessed the third number? (By multiplying the third number by 3 and

adding 5.) What will the first number be when the third number is n? ($3n + 5$.)

(10) What will the second number be? ($n - 2$.)

(11) What is the final check? (Add the first and the second numbers to see if their sum is 59.)

(12) How can you say algebraically that the sum of the first and the second numbers should be 59?

$$(3n + 5 + n - 2 = 59.)$$

(13) Solve this equation to find n, which is the third number. Then compute the other two numbers. Check to see if you are correct, just as you checked your guesses.

Your written solution should look like this:

First number	$3n + 5$
Second number	$n - 2$
Third number	n

$$\begin{aligned} 3n + 5 + n - 2 &= 59 \\ 4n + 3 &= 59 \\ 4n &= 56 \\ n &= 14 \end{aligned}$$

The third number is 14. The first is $3 \times 14 + 5$, or 47. The second is 12. CHECK. The sum of 47 and 12 is 59.

Another Analysis and Solution †

PROBLEM. At Christmas time I bought several 2-cent stamps and several 3-cent stamps, for which I paid a total of $1.60. There were fifteen more 3-cent stamps than 2-cent stamps. How many were there of each?

(1) What does the problem ask for?

To familiarize yourself with the problem and the relationships it involves, you should make a guess at the answer and check to see if you are right. Remember that guessing the right answer is not important.

(2) Will you guess both the number of 2-cent stamps and the number of 3-cent stamps? (No. I shall guess only the number of one kind; then I shall know the number of the other kind.)

You should now be able to make a guess and check to see if you are right. If you cannot, continue with the discussion.

(3) Suppose you guess that there were 20 2-cent stamps. Then how many 3-cent stamps would there be? (35.)

(4) Now find out what you had to pay for these stamps. What did you pay for the 2-cent stamps? (2×20, or $.40.) What did you pay for the 3-cent stamps? (3×35, or $1.05.) Do these values check with the problem? (No. They do not add to $1.60.)

If you do not now see clearly the relationships involved, make another guess and check its correctness. The following is the method of solving the problem with the aid of algebra.

(5) The algebraic solution follows exactly the same steps used before. If the number of 2-cent stamps is n, what is the number of 3-cent stamps? ($n + 15$.)

(6) What is the value of the 2-cent stamps? ($2 \times n$, or $2n$ cents.) What is the value of the 3-cent stamps? ($3(n + 15)$.)

(7) How can you say algebraically that the sum of these two values is $1.60? ($2n + 3(n + 15) = 160$.) Since the unit on the left side of the equation is cents, the unit on the right side must be cents, not dollars.)

(8) Solve this equation and check your answer.

Your written solution should look like this:

Number of 2-cent stamps	n
Number of 3-cent stamps	$n + 15$
Total value of 2-cent stamps	$2n$
Total value of 3-cent stamps	$3(n + 15)$

$$\begin{aligned} 2n + 3(n + 15) &= 160 \\ 2n + 3n + 45 &= 160 \\ 5n + 45 &= 160 \\ 5n &= 115 \\ n &= 23 \end{aligned}$$

The number of 2-cent stamps is 23. The number of 3-cent stamps is 38. CHECK. The value of the 2-cent stamps is $.46 and the value of the 3-cent stamps is $1.14. The sum of these two values is $1.60.

Exercises

In solving the following problems, you should follow the method discussed in the preceding pages.

1. The larger of two numbers is 5 more than 3 times the smaller number. Their sum is 102. Find the numbers.

2. The larger of two numbers is 4 times the smaller. Their difference is 25. What are the two numbers?

3. The difference between two numbers is 42. Their sum is 108. What are the two numbers?

4. A man and his son earned \$86 between them. If the father earned \$5 more than twice what the son earned, how much did each receive?

5. In triangle ABC, AB is twice as long as AC and BC is 1 inch more than AC. If the perimeter is 53 inches, how long is each side? (Draw a figure to help in the solution.)

6. Separate 75 into two parts so that one part will be 13 more than the other.

7. In a certain rectangle the length is 3 ft. more than the width. Its perimeter is 71 ft. What are the dimensions?

8. In triangle ABC, angle B is twice angle A and angle C is 3 times angle A. How many degrees are there in each angle? (Remember that the sum of the angles of a triangle is 180°.)

9. Of three numbers, the first is 3 times the second and the third is 5 more than the second. The sum of the first and third is 254. Find the three numbers.

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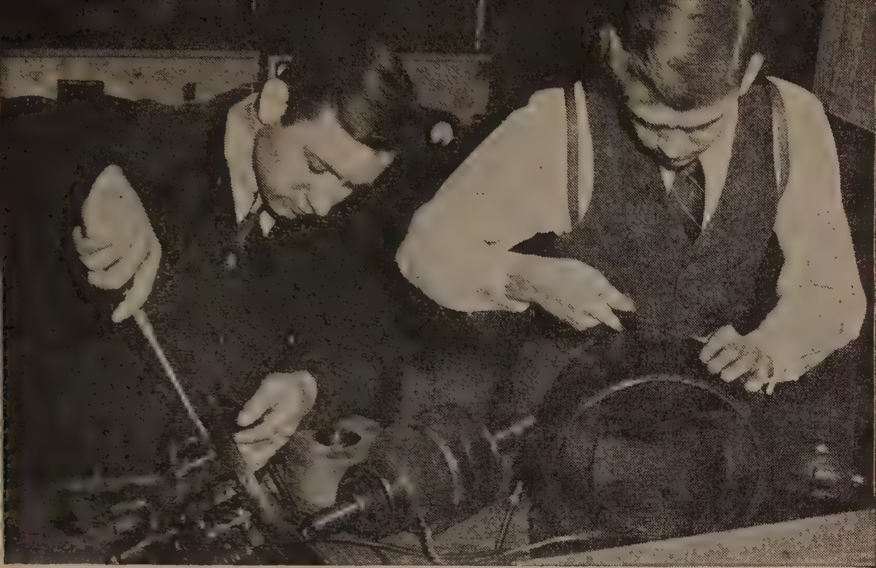
10. A debt of \$72 was paid with 5-dollar bills and 2-dollar bills, there being twice as many of the latter as of the former. Find the number of bills of each kind.

11. Two men start from the same place at the same time, one going north and the other south. One goes twice as fast as the other. The faster one travels for 7 hours and the slower for 8 hours. They are then 396 miles apart. Find the rate at which each man traveled.



Benjamin Bushey, by P. Wolff

In the shop of a high school these boys learn about electric motors. Will mathematics be of help to them when they continue in this field of work out of school?

Algebra a Help in Thinking

You are now in a position to understand that algebra is an instrument that magnifies the powers of the human mind. By the use of its methods and symbols we can analyze a problem and set down one at a time the numerical relationships the quantities bear to each other. Then we can combine our statements of relationships in an equation, solve the equation, and thus find the number values we seek. We cannot think through the entire process at once in a complicated problem, but by the use of the symbols and rules of algebra we can take the solution one step at a time and thus find the answers to problems far too complex for the unaided mind.

"How Algebra Helps Us in Thinking" would be a second way of stating our chapter theme.

To emphasize the importance that this instrument of reasoning may have for you, we ask you to carry out this exercise. Take pencil and paper and list all the occupations and professions from which you will be barred if you do not study algebra. Your list will include nearly all the professions for which college preparation is required and higher technical work in practically all fields, including the military field. Then make a second list of occupations for which algebra is not necessary but in which it is used in the advanced work. This list will include such activities as shopwork, electric work, building, business, insurance, agriculture, and a host of other common employments.

Is it evident from your lists that algebra is necessary for entrance into many lines of work and that a knowledge of it helps in rising to the higher positions in other occupations where it is not absolutely required? Evidently those who know algebra are able to do things that others cannot do.

Chapter Summary

This chapter has given you a glimpse of the possibilities of algebra in use. You have seen how rules for computation can be stated briefly and concisely by means of formulas. You made a beginning in solving equations by the use of inverse processes, and in so doing gained some idea of the value of the equation as a tool for unscrambling complex relationships and finding the value of unknowns. Using the three cases of percentage as an example, we showed you how a formula can be used as an equation to solve for any one of several quantities in it when the values of the others are given, and you were shown how algebra is used in the analysis and solution of verbal problems that are difficult or impossible by arithmetic.

You also learned about the order of operations, the meaning and use of parentheses, and the rules for multiplying an indicated sum and an indicated product by a number.

You should now know the meanings of the following technical words and phrases:

Formula
Parenthesis
Inverse processes
Distributive law
Indicated sum
Indicated product
Factor
Solving an equation
Satisfying an equation
Root of an equation
Percentage
Base in percentage
Rate in percentage
Verbal problems

Chapter Review

1. The inverses of addition, subtraction, multiplication, and division are, respectively, ___?___, ___?___, ___?___, and ___?___.

2. By merely looking at the equation $n + 4 = 9$, I see that the process to use to find n is ___?___.

3. The process to use in solving the equation $2n = 10$ is ___?___.

4. The process to use in solving $n - 3 = 7$ is ___?___.

5. The process to use in solving $\frac{n}{2} = 5$ is ___?___.

6. In solving the equation $2n + 5 = 9$, what do you do first?

7. In solving the equation $2n - 5 = 9$, what do you do first?

8. In finding the value of $8 - (4 + 2)$, what do you do first? second?

9. In finding the value of $4 + 3(9 - 4)$, what do you do first? second? third?

10. In a series of operations involving additions, subtractions, multiplications, and divisions, the ___?___ and the ___?___ should be performed first in the order in which they come and then the ___?___ and the ___?___.

11. Is 3 a root of the equation $2(x + 3) = 15$? Give a reason for your answer.

12. Does 5 satisfy the equation $3n - 7 = 8$? Give a reason for your answer.

13. What number makes the left side equal to the right side in the equation $7n + 5 - 3n = 21$?

14. When is an equation said to be satisfied?

15. What does it mean to solve an equation?

16. In the equation $2n + 7 = 16$, how many values may n have?

17. $3 + 2 + 5$ and $2a + 3b + c$ are indicated ___?___; $3 \times 2 \times 5$ and $6abc$ are indicated ___?___.

18. In the expression $2a + 3b + c$, $2a$, $3b$, and c are _?_. In $6abc$, 6, a, b, and c are _?_.

19. The expression $3(a - b)$ means 3 _?_ $(a - b)$.

20. To multiply an indicated sum by a number, I multiply _?_ term by that number. Give an example.

21. To multiply an indicated product by a number, I multiply _?_ factor by that number. Give an example.

22. To multiply $7abc$ by 2, I would multiply the _?_ by 2.

23. To multiply $2n + 5$ by 3, I would multiply _?_.

24. In the formula $p = br$, p is the _?_, b is the _?_, and r is the _?_.

25. In the formula $p = br$, to find r when $p = 300$ and $b = 1350$, I would _?_ 300 for _?_ and 1350 for _?_ and solve the resulting equation for _?_. If the result is .222, I know that the rate is _?_%.

26. Using the formula $F = \frac{9}{5}C + 32$, find F when $C = 1300°$.

27. Using the formula $C = \frac{5}{9}(F - 32)$, find C when F is 158°.

28. 18 is what per cent of 25? Give your answer to the nearest tenth of a per cent.

29. If 324 is 8% of a number, what is the number?

30. Find the value of the following:

(a) $2 \times 3 + 4$

(b) $4 + 2 \times 3$

(c) $6 + 18 \div 3$

(d) $20 - (9 - 4)$

(e) $5(6 - 3)$

(f) $15 - 2(3 + 1)$

(g) $(15 - 2)(3 + 1)$

(h) $(9 - 3) - 6$

31. Find the value of —

(a) $2(a + b)$ when $a = 3$ and $b = 2$.

(b) $(a + b)(c + d)$ when $a = 2$, $b = 3$, $c = 4$, and $d = 5$.

(c) $a - (b + c)$ when $a = 10$, $b = 5$, and $c = 3$.

(d) $a + b(c + d)$ when $a = 1$, $b = 2$, $c = 3$, and $d = 4$.

32. Multiply as indicated and combine like terms when possible:

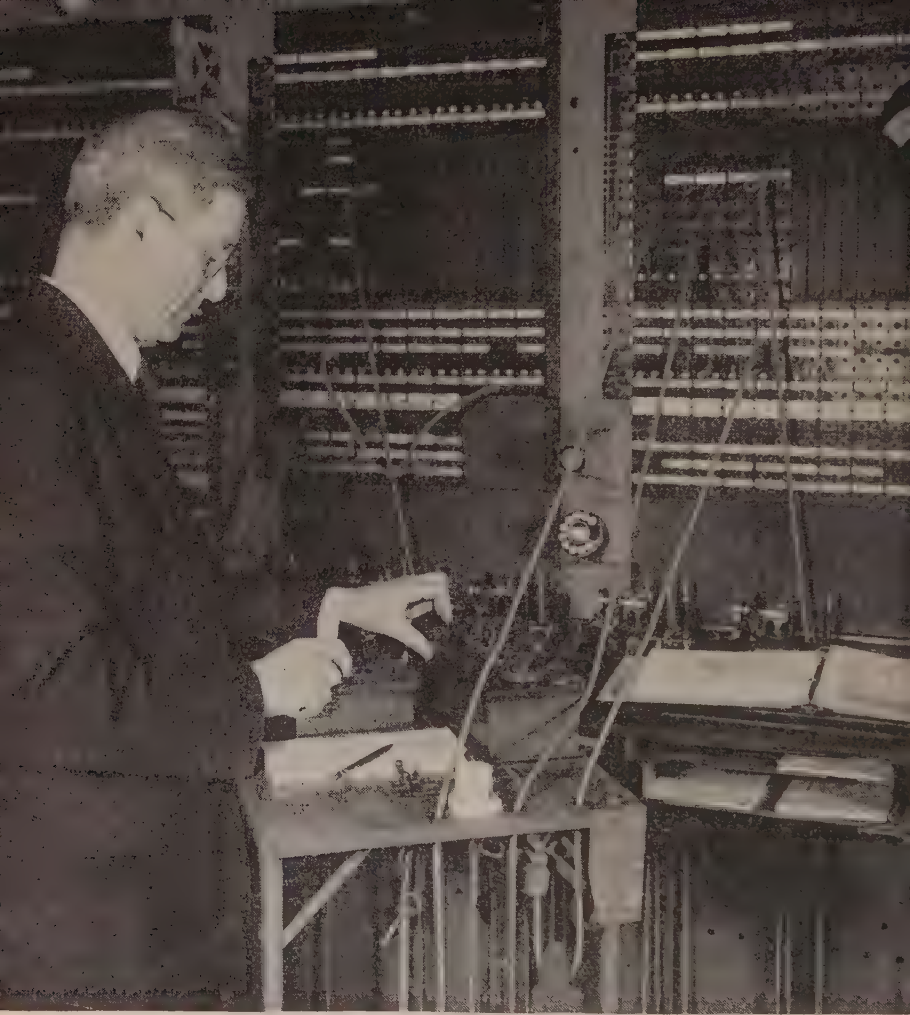
(a) $3(r + s)$
(b) $3(rs)$
(c) $6(n - 3)$
(d) $6(3\,n)$
(e) $4(2\,n + 7)$
(f) $a(b + c)$
(g) $a(bc)$
(h) $3(5\,a - 2\,b) - 6\,a$
(i) $6(4\,n) + 7(2\,n + 3)$
(j) $9 + 4(3\,n + 1)$

33. Solve the following equations and check your answers:

(a) $n + 5 = 15$
(b) $5\,n = 15$
(c) $\frac{n}{5} = 15$
(d) $n - 5 = 15$
(e) $5\,n = 7$
(f) $7\,n = 5$
(g) $3\,n + 5 = 17$
(h) $3\,n - 5 = 16$
(i) $\frac{n}{2} + 3 = 7$
(j) $5\,n + 8 - 3\,n = 17$
(k) $3(n + 2) - 6 = 9$
(l) $c + .05\,c = 6.30$

34. Henry has 3 times as many dimes as quarters. How many of each kind has he if the value of both together is $11?

35. How would you write, using an equation, that $3\,n$ is 5 more than twice the quantity $n - 1$?



Western Union Telegraph Co

When there is a break (or a fault) in a telegraph or telephone line, a man is not sent out to walk along the line to find where the break is. The distance of the break from a testing station is quickly determined with the aid of an instrument called a "Wheatstone bridge." The instrument measures the resistance to the flow of electricity in the broken wire, and by substituting the value found in a formula the distance to the break is determined.

One formula might be written $D = \frac{R_L - R_V}{2O}$, where R_L and R_V represent two measurements of resistance with the "bridge," O represents the resistance of the wire per mile, and D represents the distance in miles to the break in the wire. This is the kind of thing that can be done with formulas.

Chapter III

MAKING AND USING FORMULAS

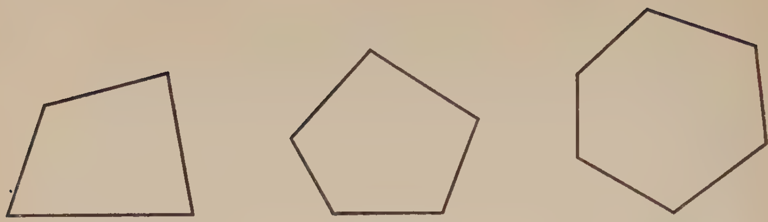
The word "formula" is from the Latin *forma*, "form." Formulas are general forms into which we fit special number values when we wish to make computations. The formulas you have used so far are simple and the formulas you will learn in this chapter will be simple; but many of the formulas used by accountants, statisticians, engineers, and scientists are much more complex. Some of them are arrived at only after years of experimentation. They are equations that express number relationships that have been tested and found true.

These formulas are to those who need to use them what the saw and the hammer are to the carpenter. They are constantly used as tools in making calculations for practical work. The strength of the framework of an ocean liner is calculated. The amount of material that must be removed for a subway or a tunnel or the amount of material needed for a road or a dam is calculated. The stresses and the strains in a bridge for given loads is calculated. The horsepower of an engine or the amount of electricity a power line can carry is calculated. The costs of all important undertakings are calculated. In all these and other calculations formulas are used.

Finding a Formula by Experiment†

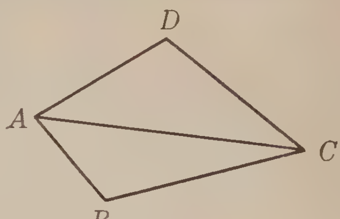
A figure formed by any number of straight lines as its sides is called a ***polygon***. A polygon of three sides is a ***triangle;*** of four sides, a ***quadrilateral;*** of five sides, a ***pentagon;*** of six sides, a ***hexagon;*** of eight sides, an ***octagon;*** and of ten sides, a ***decagon***.

What is the sum of the angles of *any* polygon?

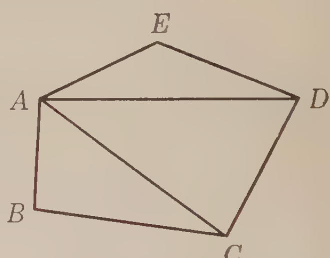


Quadrilateral Pentagon Hexagon

We shall now find a formula for the sum of the angles of a polygon of n sides by experiment. What does n mean? (Any number.)



(1) Draw a quadrilateral $ABCD$ and a diagonal AC. How many triangles are formed? (Two.) What is the sum of the angles of one triangle? (180°.) What is the sum of the angles of the quadrilateral? ($2 \times 180°$, or 360°.)



(2) Draw a pentagon and all possible diagonals from one vertex. How many triangles are formed? (Three.) What is the sum of the angles of the pentagon? ($3 \times 180°$, or 540°.)

(3) Do what you have done in Exs. (1) and (2), with polygons of 6, 7, 8, 9, and 10 sides and fill in the blanks in the following table. (Copy the table. Do not write in the book.)

No. of sides	3	4	5	6	7	8	9	10
No. of triangles . .	1	2	3	4				
Sum of angles . . .	180°	$2 \times 180°$ $= 360°$	$3 \times 180°$ $= 540°$	$4 \times 180°$ $= 720°$				

(4) What do you note about the number of triangles as compared to the number of sides? (There are two less triangles than sides.)

(5) If there were 20 sides, how many triangles would be formed by drawing all possible diagonals from one vertex? (18.) What is the sum of the angles of a polygon of 20 sides? ($18 \times 180°$, or $3240°$.)

(6) If a polygon has n sides, how many triangles will be formed? ($n - 2$.) What is the sum of the angles of a polygon of n sides? The answer is $(n - 2) \times 180$, or $180(n - 2)$.

Since n refers to any number of sides, we now have a formula for the number of degrees (D) in the sum of the angles of any polygon.

$$\mathbf{D = 180(n - 2)}$$

(7) Using this formula, find the sum of the angles of a polygon of 12 sides. (1800°.)

This method of deriving a formula is illustrative in a modest way of the methods by which scientists arrive at their formulas. By studying a large number of special cases, they arrive at a formula which is general.

Exercises

1. State in words the procedure in using the formula $D = 180(n - 2)$ when n is 6.

2. What is the sum of the angles of a polygon of 15 sides? 20 sides? 24 sides?

3. In a *regular polygon* all the angles are equal. How many degrees are there in one angle of a regular pentagon? regular hexagon? regular octagon? regular decagon?

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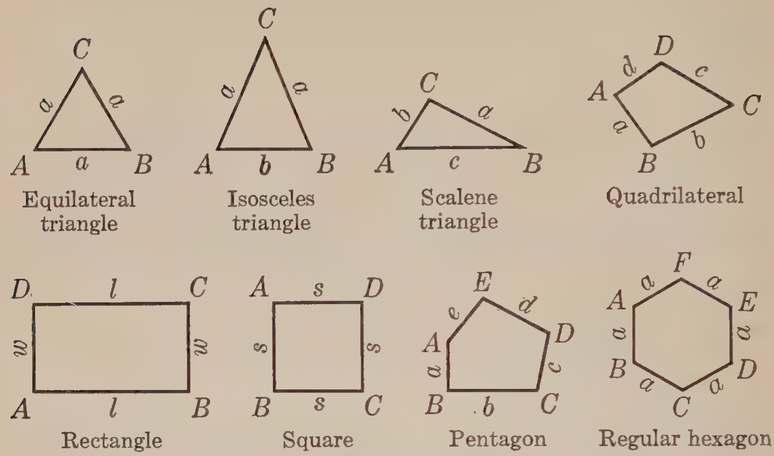
4. Write a formula for finding the number of degrees (D) in one angle of a regular polygon of n sides.

5. How many sides has a polygon if the sum of its angles is 5040? (Use the formula $D = 180(n - 2)$, substitute 5040 for D, and solve the resulting equation for n.)

6. How many sides has a polygon if the sum of its angles is 3060°?

Perimeters of Polygons



The *perimeter* of a polygon is the sum of its sides.

An *equilateral triangle* is one with its three sides equal.

An *isosceles triangle* is one which has two sides equal. In the figure, AB is called the *base.*

A *scalene triangle* is a triangle no two sides of which are equal.

A *regular polygon* is one all of whose angles and sides are equal.

In a *rectangle* the opposite sides are parallel and equal and the angles are right angles.

A *square* is a rectangle all of whose sides are equal.

Exercises

1. What is the perimeter of a scalene triangle whose sides are 15 in., 10 in., and 8 in.?

2. What is the perimeter of a square each of whose sides is 7 ft.?

3. Find the perimeter of scalene triangles whose sides are as follows:

(*a*) 6 in., 3 in., 4 in. (*c*) 5.24 ft., 6.57 ft., 3.75 ft.
(*b*) 3246 ft., 1262 ft., 2576 ft. (*d*) $6\frac{1}{2}$ in., $2\frac{1}{4}$ in., $5\frac{1}{8}$ in.

4. What is the perimeter of a regular hexagon one of whose sides is $3\frac{1}{2}$ in.?

5. Match the following formulas for perimeters and the figures with which they are associated. Use the figures on the opposite page to help you.

1. Equilateral triangle	(*a*) $p = 6a$
2. Isosceles triangle	(*b*) $p = a + b + c$
3. Scalene triangle	(*c*) $p = 4s$
4. Quadrilateral	(*d*) $p = 3a$
5. Rectangle	(*e*) $p = 2a + b$
6. Square	(*f*) $p = a + b + c + d$
7. Pentagon	(*g*) $p = 2l + 2w$
8. Regular hexagon	(*h*) $p = a + b + c + d + e$

6. Find the perimeter of equilateral triangles whose sides are as follows:

(*a*) 9 in. (*b*) $7\frac{1}{4}$ ft. (*c*) 4.8 ft. (*d*) $3x$ in.

7. Find the perimeters of isosceles triangles whose equal sides and bases are respectively as follows:

(*a*) 8 in., 5 in. (*c*) $3\frac{3}{4}$ yd., $5\frac{1}{2}$ yd.
(*b*) $2\frac{1}{2}$ ft., 3 ft. (*d*) 7.2 ft., 3.5 ft.

8. Which of these formulas is correct for finding the perimeter of an equilateral triangle: $p = a + a + a$ or $p = 3a$? Explain.

9. Which of these formulas is correct for finding the perimeter of a rectangle: $p = 2l + 2w$ or $p = 2(l + w)$? Explain.

10. What is the perimeter in feet of a rectangle whose —

(*a*) length is 8 ft. and width is 3 ft.?
(*b*) length is 4 yd. and width is 1 yd.?
(*c*) length is *a* yd., and width is *b* yd.?
(*d*) length is *p* yd. and width is *q* ft.?
(*e*) length is *c* in. and width is *d* in.?

Indirect Use of Perimeter Formulas

Study the following examples:

EXAMPLE 1. How long is the base of an isosceles triangle if one of the equal sides is 6 in. and the perimeter is 17 in.?

SOLUTION. $p = 2a + b$, $p = 17$, $a = 6$

Substituting the numbers, $17 = 2(6) + b$

$17 = 12 + b$ or $b + 12 = 17$

$b = 5$

The base is 5 inches.

EXAMPLE 2. How long is each of the equal sides of an isosceles triangle if the perimeter is 45 in. and the base is 11 in.?

SOLUTION. $p = 2a + b$, $p = 45$, $b = 11$

$45 = 2a + 11$ or $2a + 11 = 45$

$2a = 34$

$a = 17$

Each of the equal sides is 17 inches.

Exercises

1. The perimeter of a scalene triangle is 37 in.; one of the sides is 17 in. and another is 12 in. How long is the third side?

2. The perimeter of a triangle is $24\frac{1}{2}$ in.; one side is $5\frac{3}{4}$ in. and a second side is $8\frac{1}{4}$ in. How long is the third side?

3. The perimeter of an equilateral triangle is 49 in. How long is one side?

4. What is the width of a rectangle if its perimeter is 48 ft. and its length is 14 ft.?

5. What is the length of a rectangle if its perimeter is 104 in. and its width is 22 in.?

6. Using the formula $p = a + b + c$, find b when p, a, and c are as follows:

(a) $p = 12$ in., $a = 4$ in., $c = 5$ in.
(b) $p = 60$ in., $a = 20$ in., $c = 25$ in.
(c) $p = 20\frac{3}{4}$ in., $a = 9\frac{1}{2}$ in., $c = 3\frac{1}{4}$ in.
(d) $p = 18\frac{1}{3}$ ft., $a = 5\frac{1}{2}$ ft., $c = 7\frac{1}{4}$ ft.
(e) $p = 36.2$ cm., $a = 12.8$ cm., $c = 18.7$ cm.

7. Using the formula $p = 2a + b$, find a when p and b are as follows:

(a) $p = 10$ cm., $b = 2$ cm.
(b) $p = 15$ cm., $b = 4$ cm.
(c) $p = 5\frac{1}{2}$ in., $b = 1\frac{1}{4}$ in.
(d) $p = 18\frac{1}{4}$ ft., $b = 8\frac{1}{3}$ ft.
(e) $p = 62.3$ cm., $b = 20.9$ cm.
(f) $p = 14.2$ cm., $b = 8.1$ cm.

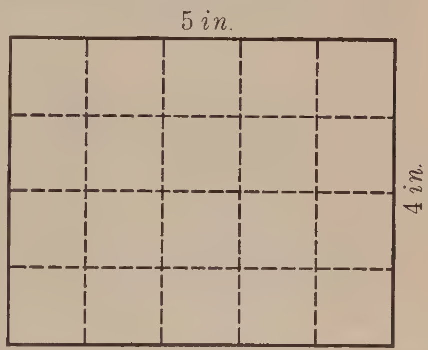
8. Using the formula $p = 2a + b$, find b when p and a are as follows:

(a) $p = 18$ in., $a = 5$ in.
(b) $p = 54$ in., $a = 20$ in.
(c) $p = 9\frac{3}{4}$ ft., $a = 4\frac{1}{3}$ ft.

Formula for Area of Rectangle

The *area* of a surface is the number of square units it contains.

(1) The rectangle at the right is 5 in. long and 4 in. wide. How many square inches does it contain? Can you answer this question without counting the squares?



(2) What did you do to the 5 and the 4 in Ex. (1) to find the number of square inches? Suppose the dimensions were 9 in. and 7 in., would you use the same process to find the area?

(3) What is the area of a rectangle whose length is l ft. and whose width is w ft.?

The number of units in the area of a rectangle equal the number of units in the length times the number of units in the width. The length and the width must be expressed in the same units.

If l represents the length and w the width of a rectangle, the area is given by the formula,

$$A = lw \qquad \text{(Area of rectangle)}$$

(4) Does the l stand for the word "length" or for a *number* of units in the length?

Exercises

1. What is the area of a rectangle whose length and width are as follows:

(*a*) 3 in., 2 in.? (*c*) 23.4 ft., 16.2 ft.?
(*b*) 18 in., 15 in.? (*d*) $4\frac{1}{2}$ ft., $2\frac{1}{4}$ ft.?

2. A sidewalk is to be 80 ft. long and 3 ft. 6 in. wide. How much will it cost to lay it at 30 cents per square foot?

3. If the area of a rectangle is 96 sq. in. and the length is 16 in., how wide is it?

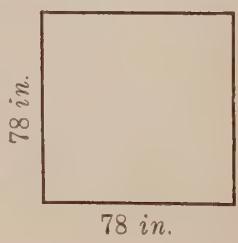
4. Using the formula $A = lw$, find w when A and l are as follows:

(*a*) $A = 24$ sq. ft., $l = 6$ ft.
(*b*) $A = 8$ sq. in., $l = 3$ in.
(*c*) $A = 5$ sq. in., $l = 6$ in.
(*d*) $A = 10\frac{3}{4}$ sq. in., $l = 5\frac{1}{2}$ in.
(*e*) $A = 8.2$ sq. cm., $l = 4.1$ cm.
(*f*) $A = 24.3$ sq. cm., $l = 5.02$ cm.
(*g*) $A = 30.25$ sq. ft., $l = 5.5$ ft.

Formula for Area of Square

As you know, a square is a *rectangle* all of whose sides are equal.

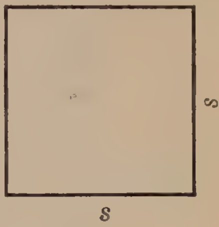
(1) Using the formula for the area of a rectangle, find the area of a square whose side is 78 in.



(2) What is the area of a square whose side is s units?

Your answer to Ex. (1) should have been 78 × 78, or 6084 sq. in. Your answer to Ex. (2) should have been $s \times s$ square units. In each case you multiplied a number by itself.

In 78 × 78, the 78 is one of two *equal factors*. In $s \times s$, the s is also one of two equal factors. The first expression can be written as 78^2, the second as s^2. They are read "78 square" and "s square." The little 2 is called an *exponent*. The exponent 2 in s^2 means that s^2 is the product of *two* equal factors, s and s.



Multiplying a number by itself — that is, using it as a factor twice — is called ***squaring the number.***

(3) What do the following mean: 5^2, 3^2, a^2, x^2?

Using this new symbolism, we can write a formula for the area (A) of a square whose side is s. It is

$$A = s^2 \qquad \text{(Area of square)}$$

Exercises

1. Square the following numbers: 1, 2, 3, 4, 5, n.

2. Find the value of the following: 6^2, 3^2, 9^2, 25^2.

3. Using the formula $A = s^2$, find A when $s = 6$; $4\frac{1}{2}$; 7.3.

4. Find the value of s^2 when s is 4; 7; 11; and 24.

5. What is the area of a square whose side is 13 ft.?

6. How many square inches are there in a square foot? How many square feet are there in a square yard? The formula $A = s^2$ helps you to get the answers.

7. Find the area of squares whose sides are as follows:

(*a*) 32 ft. (*c*) 2.1 ft. (*e*) $3\frac{1}{2}$ yd.

(*b*) 124 ft. (*d*) 3.64 ft. (*f*) $2\frac{3}{4}$ yd.

Meaning of Exponents

Many formulas contain exponents, and to operate with these formulas we need a clear understanding of what exponents are. As you have seen, the exponent 2 placed to the right and above another number means that that number is to be used twice as a factor. Similarly, the exponent 3 means that a number is to be used three times as a factor. Thus, 6^3 means $6 \times 6 \times 6$ and a^3 means $a \times a \times a$. a^3 is read "a cube." Taking a number 3 times as a factor is called *cubing* the number.

In general, the exponent n (n being a whole number) means that a number is to be used n times as a factor.

a^2 (the square of a) is also read "the second power of a" or "a to the second power." a^3 (the cube of a) is the third power of a. a^n is the nth power of a.

An *exponent* is a number placed to the right and above another number called the *base* to indicate how many times the base is to be used as a factor. Note that a^1 and a mean the same thing. The exponent 1 is usually omitted.

(1) Write $5 \times 5 \times 5$ in a simpler way.

(2) Write each of the following, using exponents: 2×2, $3 \times 3 \times 3$, $6 \times 6 \times 6 \times 6$, $s \times s \times s$.

(3) Find the value of a^3 when $a = 2$; 4; 5; and 7.

(4) Find the value of b^4 when $b = 3$.

Exercises

1. Write $b \times b \times b \times b$ in a simpler way. Write $b + b + b + b$ in a simpler way.

2. If $a = 5$ and $b = 3$, what is the value of $a^2 + b^2$? of $a + b^2$? of $a^2 + b$?

3. Find the value of the following expressions, using $a = 3$ and $b = 4$: a^2; $2a$; b^3; $3b$.

4. What is the value of p^2 when $p = 1$? when $p = 0$?

5. What is the value of each of the following? 3^2; 3^3; $2^3 \times 3^2$; $(2 \times 3)^2$; $3^2 + 3^3$; $3^2 \times 2$; 3×2^2; 1^5; 0^4.

6. To find the value of ab^2 when a is 2 and b is 3, would you multiply 2 by 3 and square the result or would you square 3 and multiply the result by 2?

7. To find the value of $(ab)^2$ when a is 2 and b is 3, would you multiply 2 by 3 and square the result or would you square 3 and multiply the result by 2?

8. The expression ab^2 means $a \times b \times b$. Tell what the following expressions mean: a^2b; a^2b^2; $2\,ab$; $2\,a^2b$; $2\,ab^2$.

9. Find the value of each of the expressions in Ex. 8 when $a = 2$ and $b = 3$.

10. What are the exponents in the expression $5\,a^3 + 3\,a^2$? What are the numerical coefficients? What are the terms?

11. If $n^2 = 25$, what is the value of n?

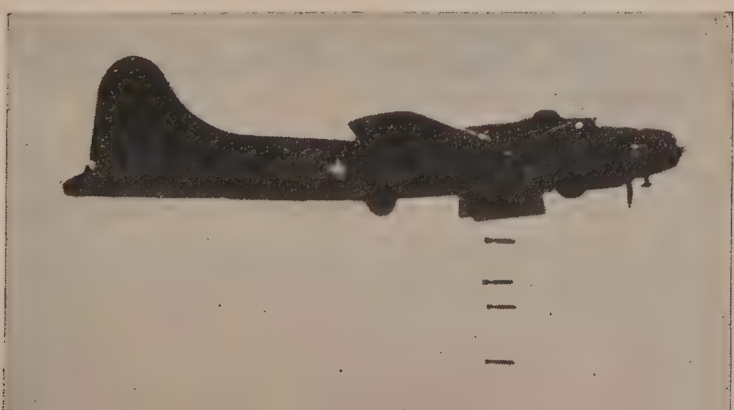
12. Find the value of the letter in each of the following equations: $a^2 = 4$; $p^2 = 16$; $b^2 = 49$.

13. Find the value of the following when $a = 5$ and $b = 2$:

(*a*) a^2	(*f*) $2\,a^2$	(*k*) $3\,b^2$	(*p*) $(ab)^2$
(*b*) b^2	(*g*) $(2\,a)^2$	(*l*) $(5\,b)^2$	(*q*) a^3
(*c*) $a^2 + b^2$	(*h*) $3\,a^2$	(*m*) ab^2	(*r*) $a^3 - b^3$
(*d*) $a^2 - b^2$	(*i*) $(3\,a)^2$	(*n*) a^2b	(*s*) $(a - b)^3$
(*e*) $(a + b)^2$	(*j*) $2\,b^2$	(*o*) a^2b^2	(*t*) $a^2 + 2\,a$

14. The number $100 = 10^2$; $1000 = 10^?$.

15. The formula $s = 16\,t^2$ tells the distance in feet that an object will fall in any number of seconds, t. In the formula, s equals the distance in feet and t the time in seconds. Find how far an object will fall in 3 seconds; in 10 seconds.



16. If a body is thrown downward with a speed of 30 feet a second, the formula $s = 16\,t^2 + 30\,t$ gives the number of feet it will fall in t seconds. How far will it fall in 6 seconds?

17. If $a = 7$ and $b = 3$, what is the value of the following?

(*a*) $a^2 - 2\,ab + b^2$ (*c*) $2\,a + 3\,b(2\,a - 3\,b)$

(*b*) $(2\,a + 3\,b)(2\,a - 3\,b)$ (*d*) $\dfrac{3\,a + 2\,b}{(a - b)^2}$

18. Find the value of the following expression when $t = 6$:

$$7\,t - 3(t + 2) + (t - 3)^2 + 5\,t^2$$

19. In a right triangle the relation between the lengths of the sides is given by the formula $c^2 = a^2 + b^2$. How much is c^2 if $a = 3$ and $b = 4$? How much is c?

20. The number 346,000 is often written as 3.46×10^5. Written this way, $5400 = 5.4 \times 10^?$.

Roots and Their Meaning

If you know that the area of a square is 25 sq. ft., can you use the formula $A = s^2$ to find the length of its side? In this case we need to find the number whose square is 25. The number whose square is 25 is called the *square root* of 25. Finding the square root of a number is the inverse of squaring a number. Thus, since the *square* of 5 is 25, the *square root* of 25 is 5. The square root of 36 is 6 because $6^2 = 36$.

Also, since the *cube* of 2 is 8 ($2^3 = 8$), the *cube root* of 8 is 2. Likewise x^2 is the second power of x and x is the square root of x^2; x^5 is the fifth power of x and x is the fifth root of x^5. Finding any root is the inverse of raising a number to any power. Do not confuse this use of the word "root" with its use as a number which satisfies an equation.

The ***square root*** of a number is one of its two equal factors. The ***cube root*** of a number is one of its three equal factors. In general, the ***nth root*** of a number is one of its n equal factors.

(1) What is the square root of 9? of 64? of 144?

(2) What is the cube root of 8? of 27? of 64?

We use the symbol $\sqrt{}$ to indicate a root. It is called a ***radical sign*** (Latin *radix*, "root"). Thus $\sqrt{4}$ means the square root of 4; $\sqrt[3]{8}$ means the cube root of 8; and $\sqrt[4]{81}$ means the fourth root of 81. The little number in the radical sign is the ***index*** of the root. The index 2 is omitted for *square* root.

(3) What is the value of $\sqrt{25}$? of $\sqrt{49}$? of $\sqrt[3]{8}$?

In the equation $x^2 = 36$, the x has been squared. To solve for x, we use the inverse process and take the square root of 36. Since $x^2 = 36$, then $x = \sqrt{36}$, or 6.

(4) Solve: (*a*) $x^2 = 100$ (*b*) $n^3 = 125$

Exercises

1. Since $5^2 = 25$, we say that 25 is the _?_ of 5 and 5 is the _?_ of 25.

2. In the formula $A = s^2$, A is the _?_ of s, therefore s is the _?_ of A.

3. What is the square root of 4, 16, 64, and 169?

4. What is the value of: $\sqrt{81}$, $\sqrt{x^2}$, $\sqrt{a^2}$, $\sqrt{d^2}$, $\sqrt{n^2}$?

5. Solve the following equations:

(*a*) $n^2 = 4$ (*b*) $n^2 = 49$ (*c*) $b^2 = 64$ (*d*) $x^2 = 25$

6. If the area of a square is 64 sq. in., what is the length of a side? (Use the formula $A = s^2$, substitute 64 for A, and solve.)

7. What is the value of $\sqrt{1}$, $\sqrt[3]{1}$, $\sqrt[4]{1}$, $\sqrt[5]{1}$, $\sqrt{0}$?

8. What is the value of: $\sqrt[3]{27}$, $\sqrt[3]{125}$, $\sqrt[3]{a^3}$, $\sqrt[3]{n^3}$?

9. Solve: (*a*) $n^3 = 1$ (*b*) $n^3 = 27$ (*c*) $n^3 = 8$

In the following expressions, combine like terms:

10. $7x^2 + 2x^2 - 5x^2$

11. $9x^2 + 2x - 3x^2$

12. $2a^2b + 3ab^2 - a^2b$

13. $x^3 + x^2 + x + 1$

14. $a(a + 2) + a^2$

15. $3n(n - 5) + 4n^2$

16. $6\sqrt{a} + 2\sqrt{a}$

17. $7\sqrt{b} - 5\sqrt{b}$

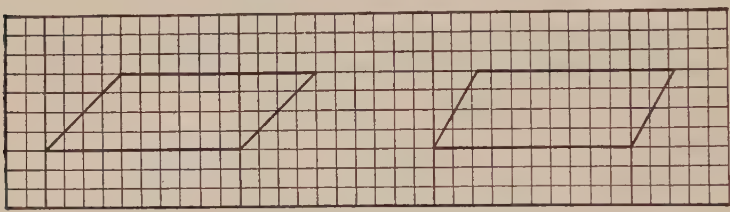
18. $7\sqrt{y} - 5\sqrt{x} + 2\sqrt{y}$

19. $6\sqrt{ab} + 3\sqrt{ac}$

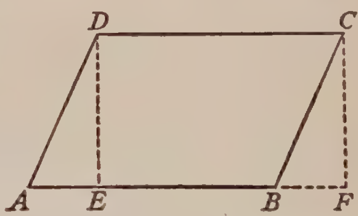
Area of Parallelogram

A ***parallelogram*** is a quadrilateral whose opposite sides are parallel. Its opposite sides are also equal.

(1) Find the area of these two parallelograms by counting the squares. What is the difficulty in this method?



(2) Draw a parallelogram $ABCD$ with the ***base*** $AB = 8$ in. and ***height*** $DE = 5$ in. Cut off the triangle AED and place it in the position BFC. What kind of figure do you have now? What are the length and the width of the new figure? What is its area? What, therefore, is the area of the parallelogram?



(3) If $AB = 10$ in. and $DE = 6$ in., what is the area of the rectangle $CDEF$? What is the area of the parallelogram? State a rule for finding the area of a parallelogram when the lengths of the base and the height are known. (The height is often called the ***altitude***.)

(4) If $AB = b$ in. and $DE = h$ in., what is the area of the parallelogram $ABCD$?

The formula for the area of a parallelogram is

$$A = bh \qquad \text{(Area of parallelogram)}$$

What do the A, b, and h stand for?

(5) Using the formula $A = bh$, find the areas of the parallelograms drawn on squared paper above. How do these answers compare with those you found in Ex. (1)? *The formula helps you to find the number of squares without counting them.*

Exercises

1. The base of a parallelogram is 9 ft. and its altitude is 4 ft. What is its area?

2. Using the formula $A = bh$, find A when b and h are as follows:

(*a*) 10 in. and 4 in.
(*b*) $13\frac{3}{4}$ ft. and $9\frac{1}{2}$ ft.
(*c*) 8 in. and 3 in.
(*d*) 32 ft. and 18 ft.
(*e*) 27 yd. and 12 yd.
(*f*) 4.8 in. and 2.3 in.
(*g*) 9.2 in. and 5.7 in.
(*h*) 3.24 in. and 1.35 in.

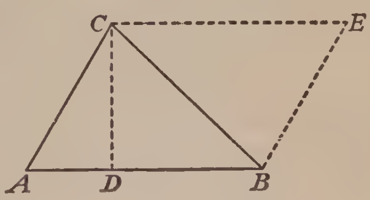
3. The area of a parallelogram is 144 sq. in. and the base is 16 in. How long is the altitude?

4. The base and the altitude of a parallelogram are equal. The area is 64 sq. ft. How long are the base and the altitude? (Let the base be n, make an equation, and solve it.)

Area of Triangle

It would be difficult to find the area of a triangle by counting the squares in it, just as it was difficult in the case of a parallelogram. The following exercises will show you how to find the area of a triangle when its base and its height are known.

(1) Cut from cardboard a triangle ABC, with base $AB = 8$ in. and height $DC = 5$ in. Cut out another triangle exactly like it and place it in the position BCE. What kind of figure is $ABEC$? What is its area?



(2) How does the base of the triangle ABC compare with the base of the parallelogram? How do the heights compare? How does the area of the triangle ABC compare with that of the parallelogram $ABEC$? What, then, is the area of triangle ABC?

(3) If the base of a triangle is 10 ft. and the height is 6 ft., what is its area?

(4) If the base of a triangle is b feet and its height is h feet, what is its area?

You see that the area of a triangle is one half the area of a parallelogram which has the same base and altitude. The formula for finding the area of a triangle with base b and altitude h is therefore —

$$A = \frac{1}{2}bh \quad \text{or} \quad A = \frac{bh}{2} \qquad \text{(Area of triangle)}$$

(5) By using several pairs of numbers for b and h, illustrate the fact that $\frac{1}{2}bh$ has the same value as $\frac{bh}{2}$.

Dividing an Indicated Product by a Number

Before you use the formula $A = \frac{bh}{2}$, you should learn the rule for dividing an indicated product by a number. (See page 49 for a similar rule.)

(1) Using $b = 6$ and $h = 4$, show that $\frac{bh}{2} = \frac{b}{2}(h) = b\left(\frac{h}{2}\right)$.

(2) Do the same thing, using $b = 10$ and $h = 6$.

(3) To evaluate $\frac{8 \times 6}{2}$, which of the following would be correct?

(*a*) Multiply 8 by 6 and divide by 2.
(*b*) Divide 8 by 2, then 6 by 2, then multiply.
(*c*) Divide 8 by 2, then multiply by 6.
(*d*) Divide 6 by 2, then multiply by 8.

Which of these procedures are correct? Which one is wrong? Which of the correct ones is easiest?

(4) Which of the ways suggested below would you use in evaluating $\frac{3 \times 14 \times 2 \times 6}{7}$?

(*a*) Divide each factor by 7 and then multiply.
(*b*) Divide 14 by 7 and then multiply.

To divide an indicated product by a number, divide only one of the factors (any one of them) by that number.

Thus, in using the formula $A = \frac{bh}{2}$, you may (1) multiply b by h and divide by 2, (2) divide b by 2 and multiply by h, or (3) divide h by 2 and multiply by b. *Do not divide both b and h by 2. If you do this, you will divide by 4.*

(5) Note also the following uses of this rule:

(*a*) $6p \div 2 = \underline{\ ?\ }$ The answer is $3p$. (We have divided the one factor 6 by 2.)

(*b*) $6p \div p = \underline{\ ?\ }$ Answer, 6. (We have divided the one factor p by p.)

(*c*) $\frac{4n^2}{n} = \underline{\ ?\ }$ Answer, $4n$. ($n^2 \div n = n$, since $n \times n = n^2$.)

(*d*) $\frac{abc}{ac} = \underline{\ ?\ }$ Answer, b. (We have divided one factor, a, by a and another factor, c, by c.)

Exercises

1. Evaluate the following in the simplest way:

(*a*) $\frac{8 \times 5}{2}$ (*d*) $\frac{9 \times 3}{3}$ (*g*) $\frac{6 \times 10}{5 \times 3}$

(*b*) $\frac{7 \times 6}{2}$ (*e*) $\frac{7 \times 3}{5}$ (*h*) $\frac{8 \times 14 \times 3}{7}$

(*c*) $\frac{7 \times 5}{2}$ (*f*) $\frac{5 \times 8 \times 3}{4}$ (*i*) $\frac{9 \times 8 \times 25}{5 \times 4 \times 3}$

2. What is the area of a triangle whose base is 8 ft. and whose altitude is 5 ft.?

3. Using the formula $A = \frac{bh}{2}$, find A when b and h are as follows:

(*a*) 4 in., 3 in. (*d*) 17.2 ft., 6 ft.

(*b*) 3 in., 8 in. (*e*) 7.8 in., 3.4 in.

(*c*) 5 in., 5 in. (*f*) 3.8 in., 6.3 in.

4. The area of a triangle is 24 sq. in. and the base is 8 in. What is the height? (Use the formula $A = \frac{bh}{2}$, substitute the numbers, and solve for h.)

5. The area of a triangle is 35 sq. ft. and the height is 7 ft. How long is the base?

6. Using the formula $A = \frac{bh}{2}$, find h when A and b are respectively as follows:

(*a*) 25 sq. in., 4 in.
(*b*) 36 sq. in., 13 in.
(*c*) 45 sq. ft., 27 ft.
(*d*) 8.2 sq. ft., 4.1 ft.

7. Divide as indicated:

(*a*) $6a \div 6$
(*b*) $8b \div 8$
(*c*) $10a \div 5$
(*d*) $15x \div 3$
(*e*) $6a \div a$
(*f*) $10a \div a$
(*g*) $15x \div x$
(*h*) $\frac{n}{n}$
(*i*) $\frac{3a}{3}$
(*j*) $\frac{3a}{a}$
(*k*) $\frac{n^2}{n}$
(*l*) $\frac{2n^2}{n}$
(*m*) $\frac{4a^2}{4}$
(*n*) $\frac{3ab}{3}$
(*o*) $\frac{6ab}{3}$
(*p*) $\frac{6ab}{a}$
(*q*) $\frac{6ab}{b}$
(*r*) $\frac{6ab}{2a}$
(*s*) $\frac{6ab}{2b}$
(*t*) $\frac{6ab}{3b}$
(*u*) $\frac{6ab}{3a}$
(*v*) $\frac{4c^2}{2c}$

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8. What is the area of a triangle whose base is $3\frac{1}{4}$ in. and whose height is $2\frac{1}{2}$ in.? (SUGGESTION. Use $A = \frac{1}{2}bh$; that is, $A = \frac{1}{2} \times \frac{13}{4} \times \frac{5}{2}$.)

9. What is the area of triangles whose bases and altitudes are as follows?

(*a*) 5 in., $2\frac{3}{4}$ in.
(*b*) $3\frac{1}{3}$ ft., $4\frac{1}{2}$ ft.
(*c*) $5\frac{1}{6}$ ft., $3\frac{1}{4}$ ft.
(*d*) $3\frac{3}{4}$ ft., $2\frac{1}{2}$ ft.

10. The base and the height of a triangle are equal. Its area is 32 sq. in. What are its base and its height?

11. If the area of a triangle is 2 sq. ft. and the height is 8 in., what is the base?

12. The base of a triangle is twice as long as its altitude (h). Express the area in terms of h.

13. The base (b) of a triangle is 3 units longer than the altitude. Express the area in terms of b.

Dividing an Indicated Sum by a Number

You must distinguish between the method of dividing an indicated product by a number and the method of dividing an indicated sum by a number.

(1) Does $\frac{a+b}{2}$ equal $\frac{a}{2}+b$ or does it equal $\frac{a}{2}+\frac{b}{2}$? Check your answer by using several pairs of values for a and b.

(2) Does $\frac{a-b}{2}$ equal $\frac{a}{2}-b$ or does it equal $\frac{a}{2}-\frac{b}{2}$? Check with $a = 8$ and $b = 6$; with $a = 14$ and $b = 10$.

The results of Exs. (1) and (2) suggest the fact that —

To divide an indicated sum (or difference) by a number, divide *every* term by that number.

See a similar rule on page 48.

(3) Study the following examples and their answers:

(*a*) $(2a + 2b) \div 2 = a + b$

(*b*) $(3r + 15) \div 3 = r + 5$

(*c*) $\frac{rs + rt + ru}{r} = s + t + u$

(*d*) $\frac{x^2 + x}{x} = x + 1$

(*e*) $\frac{ab - 3a}{a} = b - 3$

(*f*) $\frac{6a - 12b + 15}{3} = 2a - 4b + 5$

Exercises

Divide as indicated:

1. $(3a + 3b) \div 3$
2. $(3a - 3b) \div 3$
3. $9ab \div 3$
4. $(ab + ad) \div a$
5. $\frac{2n + 6}{2}$
6. $\frac{5n - 35}{5}$
7. $\frac{3r - 3s}{3}$
8. $\frac{5r + 35}{5}$
9. $\frac{35r}{7}$
10. $\frac{5n - 5}{5}$
11. $\frac{4x + 28}{2}$
12. $\frac{x^2 + 3x}{x}$

13. $\frac{3x^2}{x}$

14. $\frac{ab - 3a}{a}$

15. $\frac{3a^2b}{a}$

16. $\frac{a^2 - ab + a}{a}$

17. $\frac{2ab + 5ac}{a}$

18. $\frac{35 - 5a}{5}$

19. $\frac{9a - 18b + 27c}{3}$

20. $\frac{a^2 + ar - as}{a}$

21. $\frac{7ab + 14ac}{7a}$

Time-Rate-Distance Formula

There are always three quantities involved in any problem concerning motion; namely, the *time* (number of hours, minutes), the *rate* (number of miles per hour, feet per second), and the *distance* (number of miles, feet). The distance an object can go depends upon the rate at which it travels and the time it is traveling. You are familiar with this relationship.

Can you write a formula for the distance (d) in terms of rate (r) and the time (t) without reading further?

(1) A girl drives her car for 2 hours at an average rate of 35 miles an hour. How far does she go? What process did you use to get the distance?

(2) How far does a plane fly if it travels at the rate of r miles an hour for t hours?

If an object moves at an unchanging, or uniform, rate (r), the formula for the distance (d) it travels in a given time (t) is —

$$d = rt \qquad \text{(Time-rate-distance)}$$

You must be careful of your units. For example, if d is in miles and t is in hours, r will be in miles per hour. If d is in feet and t in seconds, r will be in feet per second. If t is in minutes and r is in miles per minute, in what units will d be?

Exercises

1. If a man walks for 4 hours at a rate of 3 miles an hour, how far can he go?

2. Using the formula $d = 35t$, copy and complete the table.

t	1	2	$3\frac{1}{2}$	5	7	8.2
d						

3. How far does an airplane go in 10.3 hours at the rate of 250 miles an hour?

4. How far will a train be from its starting point if it travels at the average rate of 43 miles an hour for h hours?

5. What is d if $r = 45$ feet per second and $t = 1$ minute?

6. If an object moves at the rate of n feet per second for n seconds, how far will it go?

7. If a man walks 10 miles in 4 hours, what is his rate?

8. If a man walks a miles in b hours, what is his rate?

9. If a train goes 180 miles at a rate of 45 miles an hour, how long was it on the way?

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10. An automobile travels for $n + 5$ hours at a rate of s miles an hour. How far, in terms of s and n, will it go?

11. Express in terms of a and r the distance covered by an automobile in a hours if its rate is —

(*a*) r miles an hour
(*b*) $4r$ miles an hour
(*c*) $(r + 3)$ miles an hour
(*d*) $(2r - 5)$ miles an hour

A Falling Object

When an object falls from a height, its rate of falling increases all the time the object is falling. Because the rate does change, the formula $d = rt$ cannot be used for a falling object.

The distance in *feet* that an object will fall in any number of *seconds* is stated by the formula $s = 16t^2$ (if we disregard its being held back by the air). You used this formula on page 73 to find the distance s when you were given the time.

Could you use this formula to find how many seconds it would take a bomb to reach the ground if it was dropped from an airplane 400 ft. high? Substituting 400 for s in the formula, we have —

$$400 = 16t^2 \text{ or } 16t^2 = 400$$

$$\text{Then, } t^2 = \tfrac{400}{16} = 25$$

$$\text{Taking square root, } t = \sqrt{25} = 5$$

It would take the bomb 5 seconds to reach the ground.

Simple Interest Formula

Money paid for the use of money is called ***interest.*** The amount borrowed is the ***principal.*** The interest is usually stated as a *per cent* of the principal for a given ***time***, commonly a year. This per cent is called the ***rate*** of interest (usually per year).

You are already familiar with interest problems. In fact, you have used the interest formula, which is the algebraic way of stating the rule.

(1) What is the interest on \$432 at 4% for 1 year?

(2) What is the interest on \$432 at 4% for 3 years?

(3) The amount of interest you will receive on \$350 for 3 years depends upon the __?__.

(4) The amount of interest you will receive on \$475 at 5% depends upon the __?__.

(5) State the rule for finding the simple interest on any principal when the time and the rate are known. Express this rule as a formula. Use i, p, r, and t.

The formula for simple interest in terms of principal, rate, and time is —

$$i = prt \qquad \text{(Interest formula)}$$

In this formula r is expressed as a decimal. If r is the rate per year (as it usually is), t must be expressed in years.

(6) Using the formula $i = prt$, find i when —

$p = \$1500$, $r = .06$, $t = 3$ yr.
$p = \$2500$, $r = .05$, $t = 6$ mo. (that is, $\frac{1}{2}$ yr.)

Exercises

1. A man wishes to borrow \$4500 to build a house. He has to pay 5% interest each year for its use. How much interest will he pay in 12 years?

2. What is the interest on \$400 at $2\frac{1}{4}$ per cent for 6 months?

3. What is the interest on \$500 at 4% for t years?

4. What is the interest on \$324 for $2t$ years at an interest rate of r (r expressed as a decimal)?

5. Using the formula $i = prt$, copy and complete the table.

p	\$650	\$362	\$185	\$1500	\$2000
r	.05	4 %	.06	3 %	$3\frac{1}{2}$ %
t	3 yr.	5 yr.	4 mo.	1 yr. 6 mo.	2 yr. 8 mo.
i					

6. Using the formula $i = .045\,p$, complete this table.

p	\$50	\$100	\$150	\$200	\$250	\$300
i						

7. On January 1 a man borrowed \$800. He paid it back on April 1 of the same year with interest at 3 %. What interest did he have to pay?

8. Mr. Laming borrowed \$560 for 1 year and paid \$22.40 interest. What was the rate?

9. Using the formula $i = prt$, find r (as a per cent) when p, i, and t are respectively as follows:

(*a*) \$230, \$27.60, 4 yr.
(*b*) \$6000, \$900, 6 yr.
(*c*) \$250, \$1.25, 3 mo.
(*d*) \$1,000,000, \$5000, 1 yr.

10. Using the formula $i = prt$,

(*a*) find t when $p = 400$, $r = .04$, and $i = 48$.
(*b*) find p when $i = 39$, $r = .03$, and $t = 2$.

---

11. What is the interest, in terms of t, on \$432 at 6 % for $t + 5$ years?

12. Mr. Wilson borrowed \$500 from Mr. Vogel at 6 % simple interest. Each year he paid the interest and at the end of 6 years he paid the principal. How much did Mr. Wilson pay Mr. Vogel during the six years?

13. Write a formula in terms of p, r, and t for the total amount (A) paid back when p dollars are borrowed at a rate r for t years?

14. Using the formula you wrote for Ex. 13, find what principal will amount to \$155 at 4 % simple interest in 6 years.

Circumference and Area of Circle

The circle is a very common figure in the practical arts, in science, and in mathematics. Much of geometry is devoted to it, and the mathematics of the circle enters largely into the calculations of the architect, engineer, navigator, geographer, and astronomer. You should know how to find the circumference (perimeter) of a circle and its area.

The circumference of a circle is about $3\frac{1}{7}$ times the diameter. This is only approximately correct, for there is no fraction or decimal by which the diameter can be multiplied to get the circumference exactly. A more accurate number than $3\frac{1}{7}$ is 3.14159. The exact value of this number is usually represented by the Greek letter π (pi). For your work in this book you may use $\pi = 3\frac{1}{7}$ or 3.14.

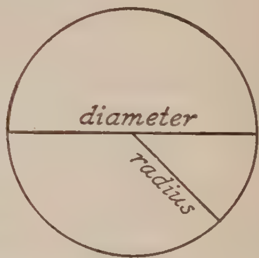
(1) The formula for the circumference of a circle with a known diameter is

$$c = \pi d \qquad \text{(Circumference of circle)}$$

Find the circumference of a circle whose diameter is 8 in.

SOLUTION. $c = \pi d;\ \pi = 3.14;\ d = 8$
$c = 3.14 \times 8$
$c = 25.12$

ANSWER. The circumference is 25.1 in. (to the nearest tenth).



Since the diameter in any circle is twice the radius — that is, $d = 2r$ — we may substitute $2r$ for d in the formula and get $c = \pi(2r)$ or $c = 2\pi r$.

(2) The area of a circle is found by multiplying the square of the radius by π. The formula is

$$A = \pi r^2 \qquad \text{(Area of circle)}$$

Find the area of a circle whose radius is 3 in.

SOLUTION. $A = \pi r^2;\ \pi = 3.14;\ r = 3$
$A = 3.14 \times 3^2$
$A = 28.26$

ANSWER. The area is 28.3 sq. in. (to the nearest tenth).

Exercises

1. Find (to the nearest tenth) the circumferences of circles having diameters as follows:

(*a*) 6 in. (*b*) 9 ft. (*c*) 18 ft. (*d*) $3\frac{1}{2}$ in. (*e*) 5.6 cm.

2. Find (to the nearest tenth) the areas of circles having radii as follows:

(*a*) 3 in. (*c*) 12 in. (*e*) $4\frac{1}{2}$ ft. (*g*) 2.1 cm. (*i*) 3.4 ft.
(*b*) 4 ft. (*d*) 5 in. (*f*) $6\frac{1}{2}$ ft. (*h*) 3.5 yd. (*j*) 8.7 cm.

3. If the radius of a circle is 8 in., how long is the diameter? If the radius is r in., how long is the diameter in terms of r?

4. If the diameter of a circle is 10 in., how long is the radius? If the diameter is d ft., how long is the radius in terms of d?

5. Find (to the nearest tenth) the circumferences and areas of circles having radii as follows:

(*a*) 2 in. (*b*) 5 in. (*c*) $2\frac{3}{4}$ ft. (*d*) 4.3 cm. (*e*) 6.2 cm.

6. What is the diameter of a circle whose circumference is 18.84 in.? (SUGGESTION. $C = \pi d$; hence $18.84 = 3.14\,d$.)

7. What are the diameters (to the nearest tenth) of circles whose circumferences are as follows?

(*a*) 56 in. (*b*) 24.2 in. (*c*) 142 ft. (*d*) 95.3 ft.

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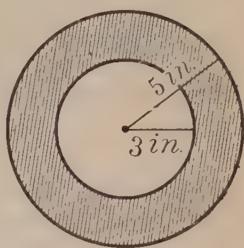
8. This ring is made by one circle inside another. The radius of the large circle is 5 in., and the radius of the small circle is 3 in. Find the area of the ring.

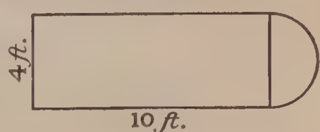


9. Write a formula for the perimeter of a semicircle (half a circle) in terms of π and d.

10. Write a formula for the area of a semicircle in terms of π and r.

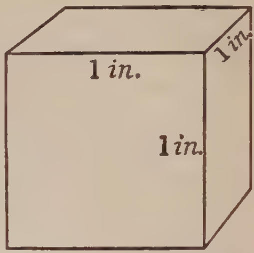
11. Write a formula for the area of a ring like that in Ex. 8, when the radius of the large circle is R and the radius of the small circle is r.

12. The figure at the right is made by placing a semicircle beside a rectangle. Find the perimeter and the area of the figure.



Meaning of Volume

When you measure a line, you use the units inch, foot, centimeter, yard, etc., and find its length. When you measure surfaces such as rectangles and triangles, you do not use these units because they do not *cover* the *surfaces.* Instead you use square units and find the area. In order to measure the space occupied by a solid, you cannot use square units because they do not *fill* the *space.* You must use such units as *cubic inches* or *cubic feet.*



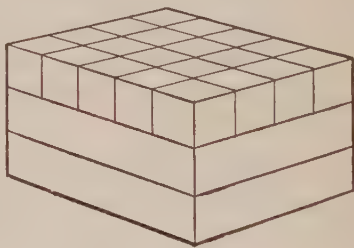
A *cubic inch* is a cube each edge of which is 1 inch. A *cubic foot* is a cube each edge of which is 1 foot. Other cubic units are cubic yards, cubic centimeters, etc.

The number of cubic units contained in a solid is called its volume.

Volume of a Rectangular Solid

The mathematical name for a solid like that represented in the drawing is *rectangular solid.* Each face of the solid is a rectangle.

(1) If a rectangular solid is 5 in. long, 4 in. wide, and 3 in. high, how many cubic inches are there in the top layer as shown in the picture? How many layers are there? How many cubic inches in all? The volume of this rectangular solid is $5 \times 4 \times 3$, or 60 cubic inches.



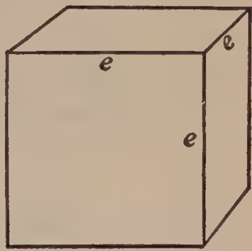
The formula for the volume (V) of a rectangular solid whose length is l units, width w units, and height h units is —

$$V = lwh \qquad \text{(Volume of rectangular solid)}$$

To use this formula, l, w, and h must be in the same units.

A *cube* is a special form of rectangular solid. It is a rectangular solid all of whose edges are equal.

(2) If each edge of a cube is 1256 in., the volume (using the formula for a rectangular solid) may be indicated as $V = 1256 \times 1256 \times 1256$. Write this in a simpler way by using an exponent.



If each edge of a cube is represented by e, the formula for the volume is

$$V = e^3 \qquad \text{(Volume of cube)}$$

(3) What does e^3 mean?

(4) What is the value of e^3 when $e = 4$?

Exercises

1. You should memorize the squares of the integers (whole numbers) from 1 to 25 and the cubes of integers from 1 to 10. What are they?

2. If a rectangular solid is 8 in. long, 5 in. wide, and 3 in. high, how many cubic inches does it contain?

3. What is the amount of air space in an empty room 15 ft. square and 9 ft. high?

4. How many cubic inches of butter are there in a piece each edge of which is 3.2 inches?

5. Find the value of l^2h, lh^2, and $(lh)^2$, when $l = 4$ and $h = 5$; when $l = 3\frac{1}{2}$ and $h = 2\frac{1}{3}$.

6. Using the formula $V = lwh$, find the volumes of rectangular solids whose dimensions are as follows:

(*a*) l = 10 ft.; w = 8 ft.; h = 6 ft.
(*b*) l = 18 in.; w = 14 in.; h = 7 in.
(*c*) l = 10 yd.; w = 8 yd.; h = 9 ft.

7. Using the formula $V = e^3$, find the volumes of cubes whose edges are as follows:

(*a*) 3 ft. (*b*) 5 in. (*c*) $\frac{1}{2}$ yd. (*d*) $2\frac{1}{2}$ in. (*e*) 2.1 cm.

8. If the length and the width of a rectangular solid are 9 ft. and 7 ft. and the volume is 315 cubic feet, what is the height?

9. If the volume of a cube is 216 cu. in., how long is one edge?

10. How many cubic feet are there in a cubic yard? (The formula $V = e^3$ will help you.)

11. How many cubic inches are there in a cubic foot? in a cubic yard?

12. How many cubes $\frac{1}{2}$ in. on a side may be placed in a cubic box whose inside edge is 8 in.?

13. Find the value of $1^2 + 2^2 + 3^2 + 4^2 + 10^2 - 5^2$.

14. Find the value of $a^2 + 2\,ab + b^2$ when $a = 5$ and $b = 3$.

---

15. A rectangular solid has a square base a units on a side and is b units high. What is its volume?

16. If the base of a rectangular solid is $l \times w$ ft. and its height is w ft., what is the volume?

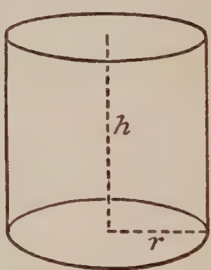
17. If the volume of a rectangular solid is 288 cu. ft. and the height is 12 ft., what is the area of the base?

18. Each edge of a cube is $2\,c$ units. What is the volume?

19. A coalbin has the dimensions l, w, and h ft. If one cubic foot of coal weighs n lb., how many tons will the bin contain?

Volume of a Cylinder

The volume of a cylinder is found by multiplying the number of units in the area of the base by the number of units in the height. This may be written $V = Bh$. Since the area of the base is πr^2 (Why?), the formula for the volume of a cylinder in terms of the height and the radius of the base is



$$V = \pi r^2 h \qquad \text{(Volume of cylinder)}$$

(1) What does $\pi r^2 h$ mean?

(2) Find (to the nearest tenth) the value of $\pi r^2 h$ when $r = 3$ and $h = 2$.

SOLUTION. $V = \pi r^2 h;\ r = 3;\ h = 2;\ \pi = 3.14$
$V = 3.14 \times 3^2 \times 2$
$V = 56.52$

ANSWER. The volume, to the nearest tenth, is 56.5.

Exercises

1. Using the formula $V = \pi r^2 h$, find (to the nearest tenth) the value of V when r and h are as follows:

(*a*) $r = 5$ in.; $h = 10$ in. (*c*) $r = 3.2$ ft.; $h = 8.1$ ft.
(*b*) $r = 3$ in.; $h = 5$ in. (*d*) $r = 4.7$ cm.; $h = 15$ cm.

2. Find the value of $2\pi r(r + h)$ when $r = 3$ and $h = 2$.

3. A cylindrical tank, the radius of whose base is 2 ft., is filled to the height of 3 ft. with gasoline. How many gallons of gasoline does it contain? (1 cu. ft. = 7.5 gal.)

---

4. By how much does a cylinder increase in volume if its height remains constantly 10 in. but its radius increases from 2 in. to 3 in.? from 3 in. to 4 in.? from 4 in. to 5 in.?

5. Frances had two cylindrical tin cans. The height of each was 6 in.; the radius of the first was 2 in., and the radius of the second was 4 in. She thought that the second can would hold just twice as much as the first. Was she right?

Writing Formulas

In many of these exercises it may help you to replace the literal numbers with numerical values, discover what processes you would use to get a numerical answer, and then indicate these processes with the letters. For example, in Ex. 2, let $l = 6$ ft. and $w = 4$ ft. The area is then 24 sq. ft. But for every square foot there are _?_ square inches; so you must _?_ 24 by _?_.

Write formulas for each of the following:

1. The perimeter (p) of a square whose side is $a + b$ units.
2. The area (A) in square inches of a rectangle, l ft. by w ft.
3. The area (A) in square feet of a rectangle l in. by w in.
4. The area (A) in square feet of a rectangle l ft. by w in.
5. The area (A) in square inches of a rectangle l ft. by w in.
6. The perimeter (p) in inches of a rectangle l ft. by w ft.; l ft. by w in.

7. The perimeter (p) in yards of a rectangle l ft. by w ft.; l ft. by w in.

8. Your average (A) in algebra if your marks for five days are a, b, c, d, and e.

9. The number of dozen (d) in n things.

10. The total cost (T) in cents of n pounds of butter at b cents per lb. and p doz. eggs at r cents a doz.

11. The cost (C) in dollars of any number (n) gallons of gasoline at c cents a gallon.

12. The number of inches (i) in a ft. and b in.

13. The number of feet (f) in a ft. and b in.

14. The distance (d) in feet that sound will travel in t seconds; in m minutes. (Sound travels 1100 feet a second.)

15. The total number of dollars (D) in d dollars, a dimes, b nickels, and c cents.

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16. The length (l) of a rectangle whose perimeter is p and whose width is w.

17. The height (h) of a triangle whose area is A and whose base is b.

18. The cost (C) of sending a telegram of n words (n being more than 10) if the cost is 25 cents up to 10 words and 2 cents for each additional word.

19. The number of tablespoonfuls (T) of coffee to use in making coffee for n persons, if the rule is one tablespoonful for every two persons and one extra for good measure.

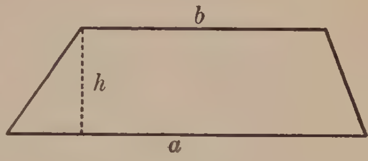
20. The cost (C) of one pencil if the list price is b cents a dozen and the discount is 10%.

21. The number of revolutions (N) a wheel will make in going a miles if the radius of the wheel is b ft. (1 mi. = 5280 ft.)

Other Formulas

1. Write as a formula: The area (S) of the surface of a sphere (ball) equals 4 times π times the square of the radius (r). Using this formula, find the area of a sphere 10 in. in diameter.

2. The formula for the area of a trapezoid is $A = \frac{1}{2}h(a + b)$, the letters having meaning as shown in the figure. What is A if $h = 5$, $a = 12$, and $b = 8$? if $h = 2\frac{1}{2}$, $a = 1\frac{3}{4}$, and $b = 2\frac{1}{4}$?



3. The formula $I = \frac{ne}{R + nr}$ is used by electrical engineers. Find I when $n = 5$, $e = 1.1$, $R = 1.2$, and $r = 1$.

4. $A = p + prt$ is a formula for finding the amount (A) that will be received from a principal (p) invested for t years at simple interest at a rate r. What will a principal of \$432 amount to in 4 years if invested at $3\frac{1}{2}\%$ simple interest?

5. The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height. Find V to the nearest tenth when $r = 8$ and $h = 10$.

6. Using the formula in Ex. 2, find b when $A = 44$, $h = 8$, and $a = 4$.

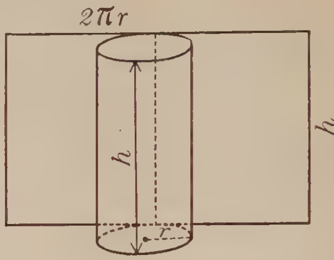
7. Using the formula in Ex. 5, find h, to the nearest tenth, when $V = 50.4$ and $r = 5.3$.

---

8. The volume of a sphere is $\frac{4}{3}$ times π times the cube of the radius. Write this statement as a formula. Using the formula, find the volume of a sphere whose radius is 6.4 in.

9. The rated horsepower (H) of a gasoline engine may be computed by the formula $H = \frac{2Nd^2}{5}$, where N is the number of cylinders and d is the diameter of each cylinder in inches. What is the rated horsepower of a 6-cylinder engine whose cylinders are $3\frac{1}{2}$ in. in diameter?

10. Divide 4724 by 23 and check your answer. Then write a formula for checking division, using D for dividend, d for divisor, Q for quotient, and R for remainder.



11. Think of a hollow cylinder made of cardboard. It can be cut on the dotted line and flattened into a rectangle as shown. The circumference of the circle becomes the length of the rectangle. If the radius of the circle is r, what is the length of the rectangle? If the height of the cylinder is h, what is the width of the rectangle? What is the area of the rectangle? What is the area of the lateral surface (the rounded surface) of the cylinder?

12. $A = P(1 + r)^n$ is a formula used in finding what P dollars will amount to (A) in n years with interest compounded annually at the rate r. What will \$452 amount to (to the nearest cent) at the end of 3 years at 4%, compounded annually?

13. A wheel which is a feet in diameter is moving along the ground without slipping. Write a formula for the distance (d) in feet that it will travel in b complete revolutions.

14. The outside dimensions of a rectangular box are l, w, and h ft. The corresponding inside dimensions are a, b, and c ft. If the material out of which the box is made weighs p pounds a cubic foot, write a formula for the weight (W) of the box.

15. A formula used in airplane designing is

$$M = \frac{2}{3}\left(a + b - \frac{ab}{a + b}\right).$$

Find M when $a = 4\frac{1}{6}$ and $b = 2\frac{1}{4}$.

16. The formula $f_x = \frac{f_4}{1 - n} + k$ is one found in a radio operator's manual. (f_x is read "f sub x" and f_4 is read "f sub 4." These symbols represent numbers just as single letters a, b, or c represent numbers.) Find the value of f_x when $f_4 = 3500$, $n = .0004$, and $k = 1$. Give your answer to the nearest tenth.

Formulas Derived from Tables

Match the following tables with the formulas given at the end:

EXAMPLE.

x	0	1	2	3	4
y	0	2	4	6	8

It is obvious that the formula is $y = 2x$, for in each case y is twice as much as x. When x is 1, y is 2; when $x = 2$, $y = 4$, etc.

1.

x	0	1	2	3	4
y	0	3	6	9	12

2.

x	0	1	2	3	4
y	1	2	3	4	5

3.

x	1	2	3	4	5
y	0	1	2	3	4

4.

x	0	1	2	3	4
y	0	5	10	15	20

5.

x	0	1	2	3	4
y	0	1	4	9	16

6.

x	1	2	3	4	5
y	1	3	5	7	9

7.

x	0	1	2	3	4
y	3	4	7	12	19

8.

x	1	2	3	4	5
y	2	9	28	65	126

9.

x	1	2	3	4	5
y	2	5	8	11	14

10.

x	0	1	2	3	4
y	3	5	7	9	11

11.

x	0	1	2	3	4
y	1	3	5	7	9

12.

x	0	1	2	3	4
y	2	4	6	8	10

13.

x	0	1	2	3	4
y	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3

14.

x	0	1	2	3	4
y	5	9	13	17	21

(*a*) $y = 3x$

(*b*) $y = x^2$

(*c*) $y = 2x + 3$

(*d*) $y = 5x$

(*e*) $y = 2x - 1$

(*f*) $y = x + 1$

(*g*) $y = x - 1$

(*h*) $y = 2x + 1$

(*i*) $y = 3x - 1$

(*j*) $y = 2x + 2$

(*k*) $y = x^3 + 1$

(*l*) $y = 4x + 5$

(*m*) $y = x^2 + 3$

(*n*) $y = \frac{x}{2} + 1$

Photo by Office of War Information

The speed of his descent depends upon his weight, the size and shape of his parachute, and the density of the air.

Dependence

A formula, as you know, states the relationship between two or more quantities. Can one of these quantities change without affecting one of the others? When the value of one of the quantities changes, how does the value of another change?

Usually a change in one thing depends upon a change in another. As the temperature changes, the height of the mercury column in a thermometer changes: the height of the mercury *depends* upon the temperature. If the radius of a circle changes, the area changes: the area of a circle depends upon the length of the radius. Upon what does the area of a square depend?

(1) Does the perimeter of an equilateral triangle change if the length of a side changes? Does the perimeter increase or decrease as the length of a side decreases? Upon what does the perimeter of an equilateral triangle depend?

(2) In the formula $A = lw$, does l change if w changes? Does A change if w changes? Does A depend upon l alone or w alone, or does it depend upon both l and w?

(3) In the formula $c = 2\pi r$, may r change? If r changes, will c change? Does π ever change in value?

Literal numbers which may take on different values are called ***variables.*** A variable whose value depends upon the value of another variable is the ***dependent variable.*** The variable (or variables) which is not dependent is the ***independent variable.***

Some quantities do not vary. Such numbers are called ***constants.*** 2 or any other arithmetic number is a constant.

In the formula $c = 2\pi r$, when you give values to r in order to find c, r is the independent variable and c is the dependent variable. 2 and π are constants, for they do not vary.

By glancing at a formula you can tell upon what the dependent variable depends. Thus, in $V = lwh$, V depends upon l, w, and h; that is, if you know the values of l, w, and h, you can determine the value of V. In $V = \frac{4}{3}\pi r^3$ (the volume of a sphere), V depends upon r. Why do we not say that V depends also upon $\frac{4}{3}$ and on π?

Exercises

1. Upon what does the cost of a roast of beef depend?

2. The number of rods of fencing needed to enclose a rectangular yard depends upon the _?_ and the _?_ of the yard.

3. The number of "movie" tickets that can be bought with a 2-dollar bill depends upon the _?_ per ticket.

4. The value of the expression $4x - 3$ depends upon the _?_ of _?_.

5. In the formula $d = rt$, I see that d depends upon the values of _?_ and _?_.

6. In the formula $A = \pi r^2$, I see that A depends upon _?_.

7. In the formula $p = 4s$, if I give values to s and then find in each case the value of p, which is the independent variable and which is the dependent variable? Is 4 a variable or a constant?

8. In the formula $p = a + b + c$, upon what does p depend? How many independent variables are there? Are there any constants?

9. If each side of a square is doubled, is the area doubled? (Draw a figure to help you.)

10. Draw a figure to show that if the length of a rectangle is trebled and the width is doubled the area is multiplied by 6.

11. Which of the following procedures would multiply the area of a rectangle by 8?

(*a*) Multiplying the length by 8.

(*b*) Multiplying the width by 8.

(*c*) Multiplying the length and width both by 8.

(*d*) Multiplying the length by 2 and the width by 4.

In Exs. 12–21 experiment with various numbers to find your answers.

12. In $p = a + b + c$, if c is increased while a and b remain constant, does p increase or decrease? If c is doubled, is p doubled?

13. In $A = lw$, what is the effect upon A if l is multiplied by 4 and w remains constant? if l is multiplied by 5 and w is multiplied by 3?

14. In $A = s^2$, does A increase or decrease as s increases? Is A multiplied by 3 if s is multiplied by 3?

15. In $M = S - C$, does M increase or decrease as C increases?

16. In $l = \frac{A}{w}$, does l increase or decrease when w increases and A remains constant? If w is multiplied by 3 while A remains constant, what is the effect upon l?

17. In $p = 2a + b$, does p increase or decrease as a increases and b decreases?

18. In $V = a^3$, what is the effect upon V when a is multiplied by 5?

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19. What would be the effect upon the interest from an investment if the principal was doubled, the rate halved, and the time trebled?

20. Which increases the size of a cylinder more, to double the height or to double the radius? ($V = \pi r^2 h$.)

21. If the radius of a sphere is multiplied by 4, is the volume multiplied by 4, 16, or 64?

22. In the formula $p = 4s$, if s varies from 2 to 5, p varies from 8 to __?__.

23. In the formula $A = s^2$, if s varies from 1 to 10, A varies from __?__ to __?__.

24. In the following formulas, if the independent variable changes as stated, how much does the dependent variable change?

(a) $V = e^3$, from $e = 2$ to $e = 3$

(b) $A = 6w$, from $w = 3$ to $w = 5$

(c) $p = 3a + 4$, from $a = 4$ to $a = 7$

(d) $V = 4l$, from $l = 1$ to $l = 3$

(e) $y = x^2 + x$, from $x = 1$ to $x = 5$

The Importance of Dependence

We study dependence because if we can discover the things upon which something else depends, we may be able to predict the second thing or possibly to control its happening. The weather depends upon several things. Since the "weatherman" knows about many of these things, he can predict with fair accuracy what the weather will be a few hours ahead. The fact that he makes a mistake occasionally shows that his knowledge of what the weather depends upon is incomplete. In contrast, the astronomer can tell to within a few seconds when an eclipse will occur hundreds of years hence. He knows almost exactly upon what it depends.

In order to control the future, the economists would like to know just exactly what makes prices change, what causes unemployment, what brings prosperity — in other words, on what these things depend. In the modern world, men are constantly on the quest for mathematical relationships between quantities which change together.

Practice in Algebraic Expression

1. Frank has five times as many dollars as James has. If n represents the number of dollars James has, what represents the number Frank has? the number they both have?

2. If a and b are two numbers, how would you represent (a) their product? (b) their sum? (c) twice their product? (d) 3 times their sum? (e) the sum of their squares? (f) the square of their sum? (g) 5 more than 3 times the first? (h) 3 less than twice the second?

3. If n represents an integer (whole number), what represents the next larger integer? the next smaller integer?

4. If one apple costs b cents, how much will c apples cost?

5. If a and b represent two numbers, how would you write one third of their sum?

6. An agent sold four times as many machines on Wednesday as he sold on Tuesday. Express in terms of the same letter the number he sold each day.

7. A class secretary was elected by a majority of 5 votes. If the unsuccessful candidate received v votes, what was the total number of votes cast in terms of v?

8. If a car runs at the rate of n miles an hour, how many miles a minute does it go?

9. A rectangular field is l ft. long and w ft. wide. What is its perimeter in yards?

10. If a oranges cost b cents, how much will one orange cost?

11. If a girl has d dollars and spends c cents, how many cents has she left?

12. What will express the number of rods of fencing required to enclose a rectangular lot which is l ft. long and w ft. wide?

13. One number is 7 times another. Let the smaller be n. (*a*) Express the other in terms of n. (*b*) Express their sum. (*c*) Express their difference. (*d*) State that the difference is 36. (*e*) Find the two numbers.

14. A man earns \$3 a day more than his son. Let the number of dollars earned in one day by the son be n. (*a*) Express in terms of n the number of dollars earned in one day by the father. (*b*) Express the number of dollars earned by both in one day. (*c*) Express the amount earned by the son in three days. (*d*) Express the amount earned by the father in five days. (*e*) How much more, in terms of n, did the father earn in five days than the son earned in three days? (*f*) State that the amount in (*e*) is \$23. (*g*) Find the value of n.

15. One number is 5 more than another. Let n represent the smaller. (*a*) Express the larger in terms of n. (*b*) Express their sum. (*c*) Express three times the larger. (*d*) Express the sum of twice the larger and three times the smaller. (*e*) State that the sum in (*d*) is 45. (*f*) Find the value of n.

16. The lengths of three sides of a triangle are such that the first side is 10 in. longer than the second side and the third side is 3 in. shorter than the second side. Let the length of the second side be n. (*a*) Express the lengths of the other two sides. (*b*) Make the statement that the perimeter is 61 in. (*c*) Solve for n and find the lengths of the three sides.

Further Work in Problem Analysis †

A father earned twice as much per day as his son. The father worked 3 days and the son worked 5 days. They received \$24.75 in all. How much did each earn per day?

In order to become familiar with the relationships involved in the problem you should make a guess at the answer and check to see if it is right. Possibly you are now able to do this without going through the following discussion. If that is the case and you are sure of the relationships, then proceed to solve the problem algebraically.

(1) Would you guess a number of dollars for both the man and the son? (No. If I know the amount for one, I shall know the amount for the other.)

(2) If you guess \$2 a day for the son, what will be the amount for the father? (\$4.) How do you know?

(3) What else must you find before you can check your answer? (The amount the father earned in 3 days and the amount the son earned in 5 days.)

(4) How much did they each earn? ($3 \times \$4$ or \$12 for the father and $5 \times \$2$ or \$10 for the son.)

(5) Do these amounts check with the given problem? (No. $\$12 + \$10$ is not \$24.75.)

Now that you see what the relationships in the problem are, you can solve it algebraically without difficulty. The quantities you must work with, as you have seen, are —

(*a*) the number of dollars earned by the son in 1 day,
(*b*) the number of dollars earned by the father in 1 day,
(*c*) the number of dollars earned in 5 days by the son, and
(*d*) the number of dollars earned by the father in 3 days.
You also know that the sum of (*c*) and (*d*) must be \$24.75.

Write down the above quantities in a column to work with, as shown below. Let the quantity (a) be represented by n and proceed just as you did with the $2 you guessed. You then have —

Number of dollars earned by the son in 1 day	n
Number of dollars earned by the father in 1 day	$2n$
Number of dollars earned by the son in 5 days	$5n$
Number of dollars earned by the father in 3 days	$6n$

$$5n + 6n = \$24.75.$$

Solve this equation and check your answer.

Exercises

1. Two boys earn together $9.75 and one boy is to receive twice as much as the other. How much will each receive?

2. The length of a rectangle is 3 times its width. If the perimeter is 78 in., what is the length and what is the width?

3. A man has a certain amount of money to pay down on a house. If he had 3 times as much, he would still lack $1399 of the purchase price, $8500. How much does he have?

4. I paid $8.75 for an advertisement of 7 lines. I contracted to pay 20 cents a line each time the "ad" appeared for the first 5 times and 5 cents a line per time after that. How many times did it appear?

5. A man divided $7000 into three parts such that the second part was twice the first part and the third part was twice the second part. How much was each part?

6. Of three numbers, the second exceeds the first by 5 and the third exceeds the second by 2. If the sum of the second and third numbers is 96, what are the three numbers?

7. The perimeter of a triangle is 47 in. The first side is 5 in. longer than the second and the third side is 3 in. less than the second. Find the length of each side.

8. One angle of a triangle is twice another. The third angle is 10° more than the smaller of the two. How many degrees are there in each angle?

9. Three boys together have 100 marbles. The first boy has 4 times as many as the second and the third has 5 times as many as the first. How many has each?

Algebra a Universal Language

Algebra has a history which reaches back many centuries before the time of Christ, and it has not always had its present finished form. The earliest algebra was a form of reckoning by complete sentences. Gradually a few symbols were introduced for often-repeated ideas, and in the sixteenth and seventeenth centuries symbolic algebra had its great development.

Now we have an algebraic language in which, as you have seen, it is possible to make many statements briefly and with absolute exactness. In this language a formula of three letters may tell us as much as a long sentence, and an equation may explain more than a paragraph. One great advantage of algebra is that it can say so much with so few words.

A further advantage of the language of algebra is that it is used universally. When the mathematicians of the world meet, they may have difficulty in understanding each other's words; but the symbols, formulas, and equations of algebra make up a language common to them all.

A very considerable portion of the world's knowledge, especially its scientific knowledge, is recorded in this universal language of mathematics, and it is a decided advantage to be able to read it. You should learn the elements of algebra not only because through all the changing conditions of many centuries it has proved its value, but also because, like music, literature, and the arts, it is recognized by educated persons of all nations as an important element in universal culture and a prized heritage from the past.

> **Remarque.** — Il résulte, immédiatement, de la définition de la somme de deux nombres, que
>
> $$a + b = b + a,$$
>
> car rien dans la définition n'indique quel est celui des deux nombres qu'on écrit le premier et une somme arithmétique de deux nombres arithmétiques ne dépend pas de l'ordre de ces nombres.

The word language of France is different from that of the United States, but in both countries the language of algebra is the same. The illustration above shows a paragraph from an elementary French algebra.

Chapter Summary

In this chapter you found one formula by experiment as a very simple illustration of the method by which formulas are derived by scientists. You had practice in the direct and the indirect use of several formulas. If you know the value of all but one letter in a formula, you can find the value of the unknown letter (by solving an equation if necessary). And you had practice in writing formulas.

In addition you learned the following:

Meaning and use of exponents in indicating powers of numbers. Taking roots of numbers is the inverse of raising them to powers.

Meaning of square root, cube root, and roots in general, with the symbols for them.

How to divide an indicated product and an indicated sum by a number.

Meaning of dependence. The way certain quantities depend upon others and the way one quantity changes as another changes.

You should now know the meaning of the following technical terms:

Polygon
Regular polygon
Quadrilateral
Pentagon
Hexagon
Octagon
Decagon
Scalene triangle
Isosceles triangle
Equilateral triangle
Rectangle
Square
Base of isosceles triangle
Area

Exponent
Square of a number
Cube of a number
Power of a number
Square root of a number
Cube root of a number
Radical sign
Index of a root
Parallelogram
Altitude of a parallelogram
Altitude of a triangle

Circle
Circumference
Radius
Diameter
Volume
Rectangular solid
Cube
Cylinder
Variable
Independent variable
Dependent variable
Constant

You should know the following formulas:

Sum of the angles of a polygon	$D = 180(n - 2)$
Perimeter of an equilateral triangle	$p = 3a$
Perimeter of an isosceles triangle	$p = 2a + b$
Perimeter of a scalene triangle	$p = a + b + c$
Area of a rectangle	$A = lw$
Area of a square	$A = s^2$
Area of a parallelogram	$A = bh$
Area of a triangle	$A = \frac{1}{2}bh$ or $\frac{bh}{2}$
Time-rate-distance formula	$d = rt$
Simple interest formula	$i = prt$
Circumference of a circle	$c = 2\pi r$
Area of a circle	$A = \pi r^2$
Volume of a rectangular solid	$V = lwh$
Volume of a cube	$V = e^3$
Volume of a cylinder	$V = \pi r^2 h$

Chapter Review

1. Write the following, using exponents:

(*a*) The second power of a
(*b*) The square of 5
(*c*) The cube of 2
(*d*) The fourth power of $(a + b)$

2. Using a radical sign, indicate the following:

(*a*) The square root of 25
(*b*) The square root of a
(*c*) The cube root of 64
(*d*) The fourth root of $3x$

3. Write the following more simply:

(*a*) $b + b + b + b$
(*b*) $a \cdot a \cdot a \cdot a$
(*c*) $5 \cdot 5 \cdot 5$
(*d*) $5 + 5 + 5$

4. What is the value of 3^4, 2^5, 5^3, and 1^5?

5. When n is 4, what is the value of n^3? of $3n$?

6. What is the 4 in n^4 called? What does it indicate?

7. What is the 4 in $\sqrt[4]{81}$ called? What does it indicate? What is the value of $\sqrt[4]{81}$?

8. What is the value of $\sqrt{36}$, $\sqrt[3]{64}$, $\sqrt[5]{32}$, $\sqrt{n^2}$?

9. Solve the equations: (*a*) $n^2 = 49$ (*b*) $\sqrt{n} = 16$ (*c*) $a^3 = 64$

10. Find the value of the following when $a = 3$ and $b = 2$:

(*a*) a^2 (*c*) $2a^2$ (*e*) a^2b (*g*) $(ab)^2$ (*i*) $(a + b)^2$
(*b*) b^3 (*d*) $2b^3$ (*f*) ab^2 (*h*) ab^3 (*j*) $a^2 + b^2$

11. When x is 4 and y is 3 what is the value of $\sqrt{x^2 + y^2}$?

12. State the difference between the methods of dividing an indicated product by a number and dividing an indicated sum by a number.

13. Divide as indicated:

(*a*) $14a \div 7$

(*b*) $(14 + 7a) \div 7$

(*c*) $\frac{14a}{a}$

(*d*) $\frac{12ab}{3b}$

(*e*) $\frac{4a + 6b}{2}$

(*f*) $\frac{2a^2 + 4ab}{a}$

14. Write formulas for the following:

(*a*) Area of a rectangle
(*b*) Area of a square
(*c*) Area of a parallelogram
(*d*) Area of a triangle
(*e*) Circumference of a circle
(*f*) Area of a circle
(*g*) Volume of a rectangular solid
(*h*) Volume of a cylinder

15. What is the perimeter of an isosceles triangle whose base is 5 in. and whose equal sides are each $6\frac{1}{2}$ in.?

16. If the perimeter of an isosceles triangle is 26 in. and the base is 9 in., how long is each of the equal sides?

17. If the perimeter of a rectangle is 41 in. and the length is 12 in., what is the width?

18. What is the area of a triangle whose base is 5.7 ft. and whose height is 3.2 ft.?

19. What are the circumference and the area of a circle whose radius is 7.2 in.?

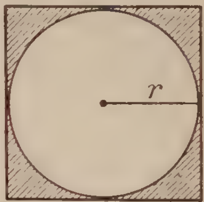
20. What is the interest on \$536 at $2\frac{1}{4}\%$ for 5 years?

21. How many degrees are there in the sum of the angles of a polygon of 15 sides?

22. How many degrees are there in one angle of a regular polygon of 20 sides?

23. How many sides has a polygon if the sum of its angles is 2160°?

24. What is the volume of a cylinder whose radius is 3.6 ft. and whose height is 4.2 ft.?



25. If the area of a triangle is 54 sq. in. and the base is 36 in., what is the height?

26. The figure at the right shows a circle inside a square. Write a formula for the area (A) of the shaded surface in terms of the radius (r) of the circle.

27. In the formula $V = \frac{1}{3}\pi r^2 h$, the formula for the volume of a cone in terms of its radius and its height, name the variables and the constants. Why are they called "variables"?

28. When the length of each side of a square is doubled, by what number is the area multiplied?

29. Does $a + \frac{b}{c}$ increase or decrease as c increases?

30. How many units does $x^2 + 3x + 2$ increase as x increases from 2 to 5?

31. Using the formula $F = \frac{9}{5}C + 32$, find C when $F = 40°$.

32. Using the formula $A = P(1 + rt)$, find P when $r = .04$, $A = 260$, and $t = 2$. Give the answer to the nearest hundredth.

33. 27 is what per cent of 63? Give your answer to the nearest tenth of a per cent.

34. Write a formula for computing wages (W) at 90¢ an hour for 40 hours and time and a half for overtime. Use r for number of hours up to and including 40 and t for number of hours overtime.

Maintaining Skills[1]

(Fundamental Operations with Integers)

As you continue your study of algebra you should keep up your skills in handling arithmetic numbers. The following will give you practice in the fundamental operations with integers.

Add:

1. 5930	**2.** 9820	**3.** 3948	**4.** 9736
9474	4679	2490	6488
2967	5685	3329	6079
9506	9530	3004	7227
3035	6677	6302	9327

Subtract:

5. 24249	**6.** 78053	**7.** 927086	**8.** 10000
23776	9980	42987	3246

Multiply:

9. 5761	**10.** 9516	**11.** 8324	**12.** 7634
39	46	500	207

Divide:

13. $5\overline{)6723}$ **14.** $6\overline{)8361}$ **15.** $7\overline{)7342}$

16. $4\overline{)2036}$ **17.** $48\overline{)59232}$ **18.** $94\overline{)19458}$

19. $65\overline{)65200}$ **20.** $52\overline{)2367}$

Multiply as indicated:

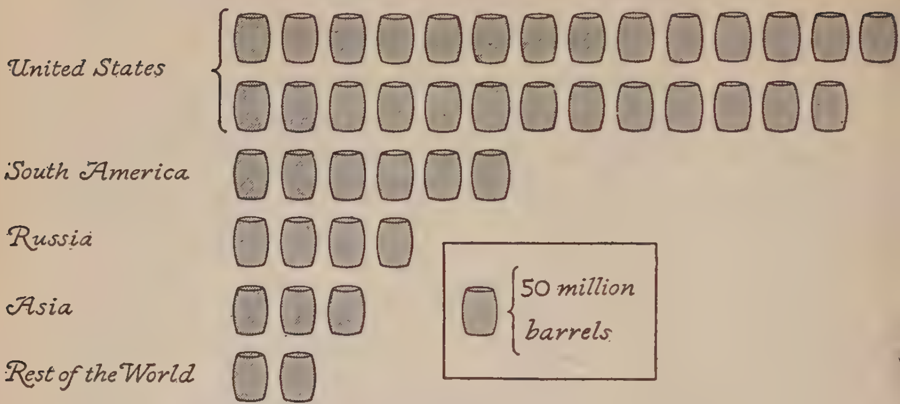
21. 23×10 **22.** 46×100

23. 492×1000 **24.** 642×10

25. In a certain army camp the soldiers allotted on the average $9.75 a month from their pay for the purchase of War Bonds. Find the total amount allotted in a year by a company of 250 men.

[1] To the Teacher. See Note 10 on page 460.

WORLD PRODUCTION OF PETROLEUM



American Petroleum Institute

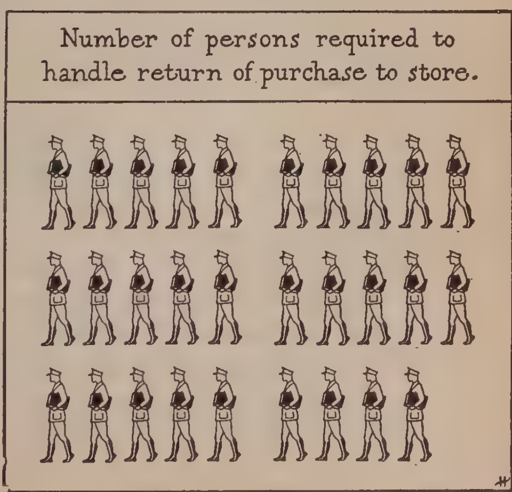
Chapter IV

HOW TO PICTURE NUMBERS — GRAPHS

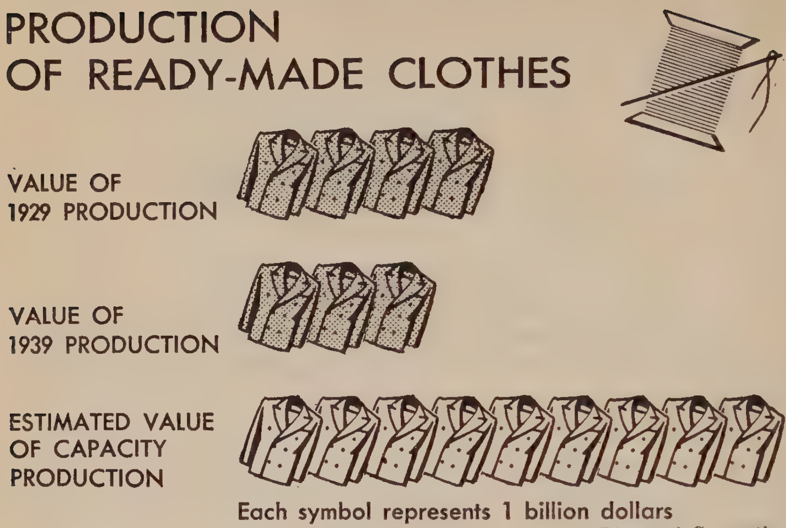
You have seen numbers pictured in newspapers and in magazines. In an article concerning the amounts of petroleum produced by different countries, for example, there will be a diagram showing oil drums — so many drums for this country, so many for that — stretched out in rows. The comparative lengths of these rows — that is, the comparative numbers of drums in the rows — help you to understand and retain the numbers you read in the article. You do not think of these little pictures as oil drums; you think of them as numbers. Each drum represents to you perhaps *fifty million* barrels of oil. This is one way of picturing numbers. Such a diagram is called a *pictogram.*

Pictograms

When a number (a whole number) is small, it can be pictured by drawing that number of things just as primitive men did it. The pictogram below was put out by a department store to urge its customers to avoid unnecessary returns of purchases. Each figure represents *one* man.



When numbers are large, they cannot be pictured by drawing that many symbols. Each symbol in the pictogram must then represent more than one



Pictograph Corporation

thing — five, ten, a hundred, sometimes a million, or a billion things (of the same kind).

In the pictogram above you see several coats. Does the picture mean to convey to you anything about coats? The answer is "No." Each coat is a symbol for a number. It represents a *billion* dollars. When you look at the top row, you do not think "four coats"; you think "four *billion* dollars."

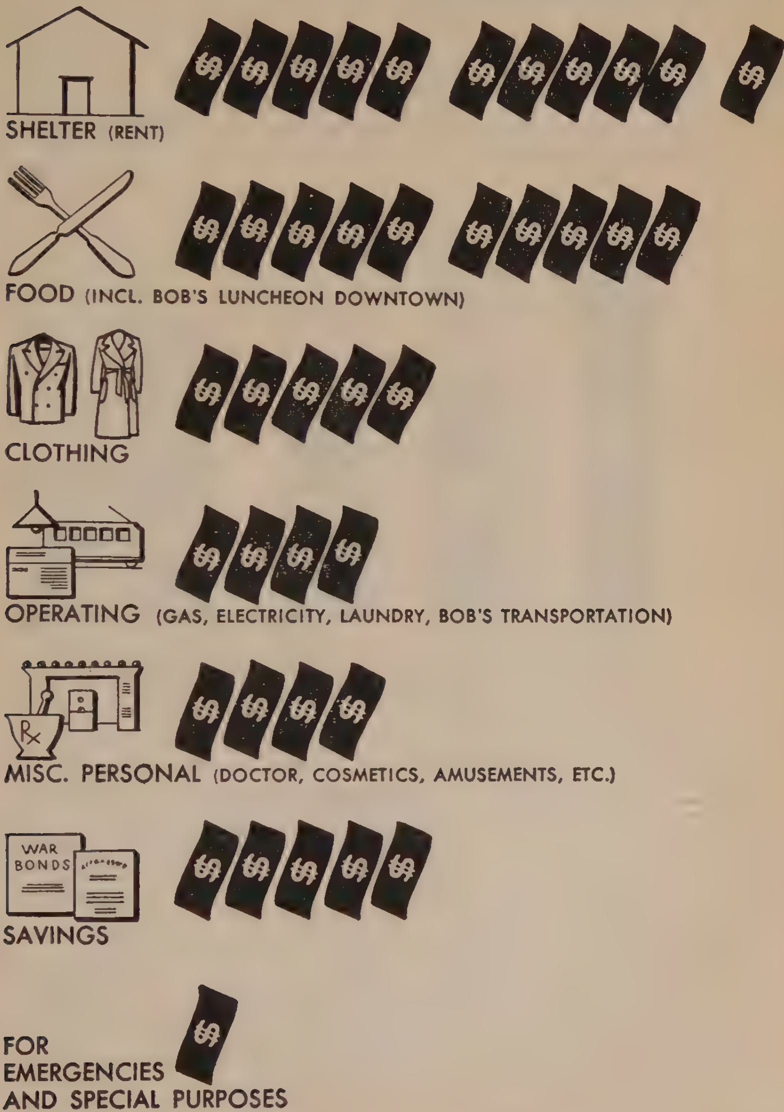
Similarly, the nine coats in the lower row represent nine billion dollars, the value of the ready-made clothing that would be produced if the industry was using all its available equipment and men.

Exercises

1. Interpret the pictogram on page 113. (a) For what was the largest amount of money budgeted? (b) For what was the next largest amount budgeted? (c) For what was the smallest amount budgeted? (d) What does each symbol represent? (e) How much was budgeted for each type of expenditure? (f) Are you expected to think *five-dollar bills* or *number* of dollars when you look at this pictogram?

2. Be sure you can interpret the pictogram on page 111.

THE WILSONS' BUDGET
($200 PER MONTH)

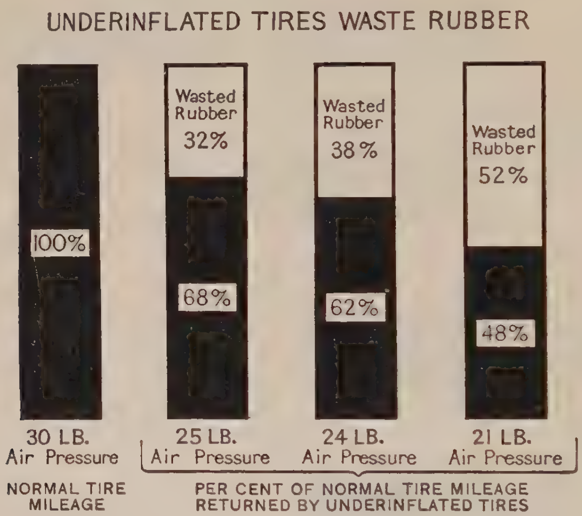


Each symbol represents 5 dollars

Pictograph Corporation

Representing Numbers by Lengths of Lines

You are also familiar with *bar graphs* and *line graphs*. In them numbers are represented by lengths of lines. When you see such graphs, if you interpret them correctly, you do not think of the lines themselves but of the numbers they represent. Just as you have learned to think of the letters in algebra as numbers, *you must learn to think of the lengths of lines in graphs as numbers.*



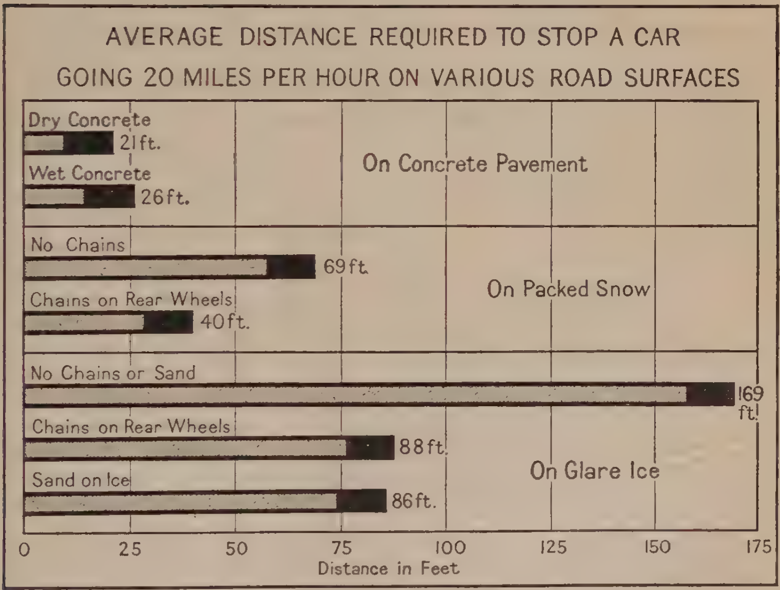
(1) Look at the graph above. Here numbers are represented by lengths of lines. For example, 48 per cent is represented by the black bar which is just a little less than half the length of the black bar for 100 per cent. The width of all the bars is the same; the only important feature is their comparative lengths. As you look at the graph, you see first a black bar about an inch and a half long. However, you are expected to think, not of the actual length of the bar, but of the number it represents in comparison with the numbers represented by the lengths of the other bars. Notice that the comparison of these lengths may be made very easily because they are all measured from a common base.

(2) Of the lines at the right, the first may represent one horse, one man, forty airplanes, a thousand pounds of sugar; in fact, *any number* of things you wish it to represent. The second line is three times as long as the first; hence it represents three times as much.

(3) If a line $\frac{1}{4}$ in. long represents 10 men, what does a line 1 in. long represent?

(4) If a line 1 unit in length represents $100, what would represent $175?

The next graph also is made on the principle that a length of line may represent a number. Here a certain length of line represents 25 ft.; a line twice as long represents 50 ft.; and so on. Note the scale at the bottom to help in reading the graph.

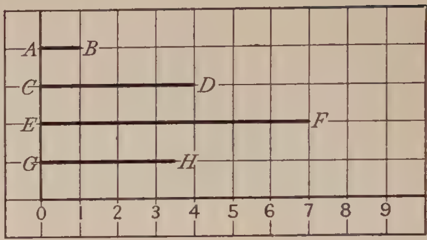


(5) How would you represent a distance of 80 ft. on this graph?

This method of picturing numbers (by length of line) is the one most frequently used in algebra. It is the basis of the graphs you will study and make.

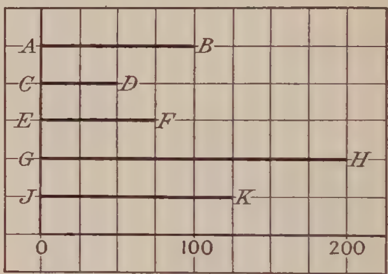
Exercises

1. *AB*, which is one unit in length (see scale at the bottom of the figure), represents six dollars. What do *CD*, *EF*, and *GH* represent?



2. On a piece of graph paper let one unit of length (a side of one square) represent 200 armored trucks. Draw lines representing 400, 800, 900, and 350 trucks.

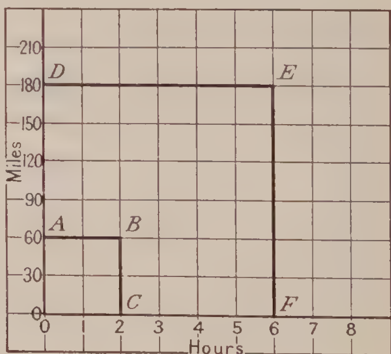
3. *AB* represents 100 school children. What do *CD*, *EF*, *GH*, and *JK* represent?



4. On a piece of graph paper let one unit of length (a side of one square) represent one foot. Draw lines representing 2, 6, 10, 12, 7, and $9\frac{1}{2}$ feet.

5. On a piece of graph paper, let one unit of length represent the number 8. Draw lines representing 16, 24, 32, 12, 20, and 4. How long a line will represent the number *zero?*

6. On this chart a horizontal line represents a number of hours and a vertical line represents a number of miles. A horizontal unit of length is 1 hour and a vertical unit of length is 30 miles. (See scales.) How many units long is *AB*? What does it represent? How many units long is *CB*? What does it represent? What do *DE* and *FE* represent?



Representing Two Related Numbers[1]

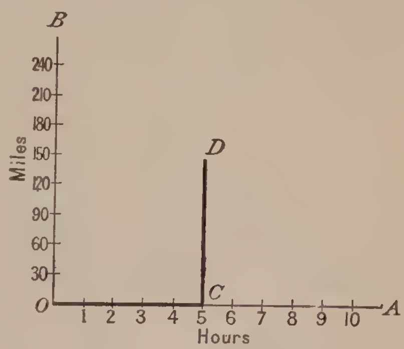
If I travel for several hours at the rate of 30 miles an hour, the distance d I shall cover in any given number of hours t is given by the formula $d = 30t$. From this formula I can make the following table:

When t is	0	1	2	3	4	5	6	7
then d is	0	30	60	90	120	150	180	210

This table is read: When t is 0, d is 0; when t is 1, d is 30; when t is 2, d is 60; etc. The 2 and 60 are *related numbers;* they go together. So are 0 and 0, 1 and 30, 4 and 120. We shall call them ***number-pairs*** and write them (0, 0), (1, 30), (2, 60), etc.

What we have learned so far about representing numbers by lengths of lines gives us a basis for representing these related pairs of numbers, these number-pairs.

To begin with we draw two perpendicular lines (lines at right angles to each other), such as OA and OB. OA is called the ***horizontal axis*** and OB is the ***vertical axis.*** The point O is the ***origin.*** We can then represent any number of hours along the horizontal axis and any number of miles along the vertical axis by using suitable units on each.



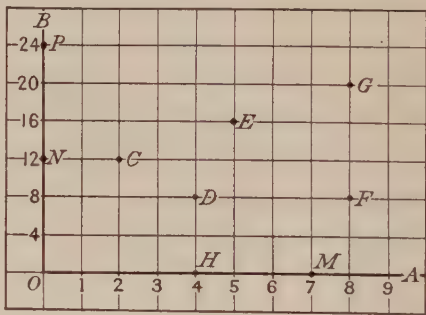
Let one horizontal unit of length represent 1 hour and one vertical unit represent 30 miles. To represent the number-pair (5, 150), meaning *in 5 hours I go 150 miles* (see problem and table), start at O and measure 5 units to the right. This distance, OC, from the vertical axis (measured perpendicular to it) represents the 5 hours.[1] From C, measure up 5 units (each unit representing 30 miles). The perpendicular distance, CD, from the horizontal axis represents 150 miles. The point D then represents the pair of numbers (5, 150).

[1] To the Teacher. See Note 11 on page 460.

In representing a pair of related numbers, we shall use the horizontal axis to represent the first number of the pair, the independent variable (see page 97), and the vertical axis to represent the second number, the dependent variable. A point located by the lengths of perpendicular lines from it to the two axes will then represent the number-pair. Graph paper helps, of course, in measuring the lengths of the lines which locate a point.

(1) On a piece of graph paper draw the axes OA and OB and write in the horizontal and vertical scales as was done on page 117. Represent by points the number-pairs (1, 30), (2, 60), (3, 90), (4, 120), (5, 150), (6, 180), (7, 210). What point represents the number-pair (0, 0)? Locating the points is called *plotting the points.*

(2) What number-pairs are represented by points $G, D, E, F, C, O, H, M, N$, and P on the accompanying graph chart? (Note that one unit of length measured horizontally represents 1 and one unit vertically represents 4.)



Exercises

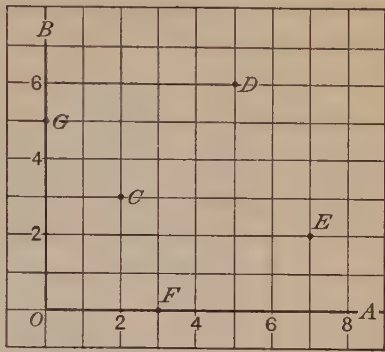
1. The following table is derived from the formula $A = 6h$. Check its correctness. On a piece of graph paper draw axes OA and OB. Let the horizontal unit represent 1 and the vertical unit represent 3. Then plot (locate) the points which represent the number-pairs in the table; that is, (0, 0), (1, 6), (2, 12), etc.

When h is	0	1	2	3	4	5	6	7	$7\frac{1}{2}$
A is	0	6	12	18	24	30	36	42	45

2. Using one unit to represent 1 both horizontally and vertically, plot the points which represent these number-pairs:

(*a*) (2, 1) (*c*) (8, 3) (*e*) (8, 0) (*g*) (7, 3)
(*b*) (5, 7) (*d*) (0, 0) (*f*) (0, 7) (*h*) (0, 2)

3. On a graph chart choose suitable scales for the horizontal and vertical axes and plot the point (5, 100). Make heavy the lines which you have chosen to represent the distances from the axes, 5 and 100.



4. What number-pairs are represented by the points *C*, *D*, *E*, *O*, *F*, and *G* in the chart at the right?

5. If $p = 2a + 4$, what is the value of p when $a = 5$? when $a = 0, 1, 2, 3, 4, 5, 6$, and 7, respectively? Make a table showing these related pairs of numbers. Plot the points representing these number-pairs.

6. Make a table of related values for the formula $p = 3a + 1$, using the values $a = 0, 1, 2, 3, 4$, and 5. Represent the related pairs of numbers on a graph chart.

7. Make a table of related values for the formula $A = s^2$, using the values $s = 0, 1, 2, 3, 4$, and 5. Represent the number-pairs on a graph chart.

Picturing a Formula

What you have learned thus far in this chapter makes it easy for you to picture a formula (or equation) with two variables. The picture is called the *graph of the formula* (or equation).

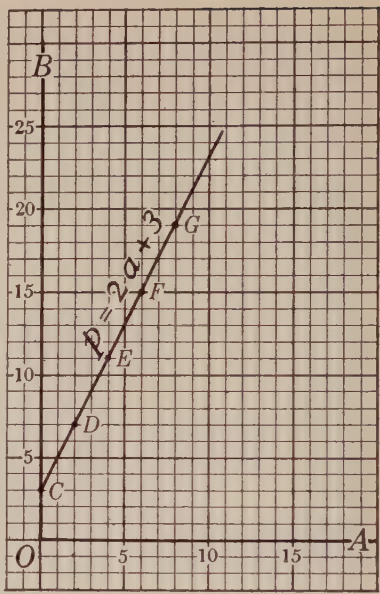
Example. Draw the graph of $p = 2a + 3$.

(1) Make a table of related values of a and p.

a	0	2	4	6	8
p	3	7	11	15	19

(2) Draw the axes *OA* and *OB* at right angles to each other.

(3) Choose suitable scales for the two axes. As we have said, the scale for the independent variable is usually made on the horizontal axis.



(4) Locate the points (0, 3), (2, 7), (4, 11), etc. (Remember, (0, 3) is a short way of writing $a = 0$, $p = 3$.) This gives points C, D, E, F, and G.

(5) The *line* through the points C, D, E, F, and G *is the graph* of $p = 2a + 3$. Continue the line beyond G to show that it may continue indefinitely. Write the formula $p = 2a + 3$ on the chart. Notice that the graph here is a *straight line* through the points. You will find other graphs later where the line through the points is a *curved line.*

(6) Using the graph (that is, the line), find p when $a = 3$. (Starting at 0, go 3 units to the right. From this point go straight up until you meet the graph. You will have to go up 9 units. Therefore, p is 9 when a is 3.)

(7) Using the graph, find a when p is 10. (This time start at 0 and go up 10 units. How many units must you go to the right before reaching the graph?)

Steps in Graphing a Formula

1. Make a table of related values of the variables.
2. Draw horizontal and vertical axes on graph paper.
3. Choose suitable horizontal and vertical scales. Indicate the scales on the chart.
4. Locate (plot) the points which represent the pairs of related numbers in your table.
5. Draw a line through the points. This line may be straight or curved, depending upon the formula or equation being plotted.
6. Write on the chart the formula which the graph represents.

Exercises

1. The following table was made by using the formula $p = 3s$.

s	0	3	6	9
p	0	9	18	27

Draw a pair of axes, plot the points corresponding to the pairs of related numbers in the table, and draw the graph. Using the graph, find p when $s = 5$. Find s when $p = 21$.

2. Draw the graphs of the following formulas:

(*a*) $A = 3h$
(*b*) $p = 3a + 5$
(*c*) $r = 4s + 1$
(*d*) $F = 7w$
(*e*) $C = 40n$
(*f*) $d = 20t$
(*g*) $N = c + 1$
(*h*) $p = 3n - 6$

For (*h*) use the values $n = 2, 4, 6, 8$, and 10.

---

3. Draw the graph of the formula $C = \frac{9}{5}F + 32$. (Use $F = 0, 5, 10, 15$, and 20.)

4. On a large piece of graph paper draw the graph of $A = s^2$, using $s = 0, 1, 2, 3, 4$, and 5. (For the horizontal scale use one large unit (ten small units) to represent 1, and for the vertical scale use one large unit to represent 5.) Note that the graph here is not a straight line. To make this graph, draw a *curved line* through the points.

Dependence Shown by Tables and Graphs

In Chapter III you studied dependence as it is shown in formulas. It may also be shown by tables and graphs. The interest you must pay for the use of a given amount of money at a given rate of interest depends upon the time you keep it. The bank clerk uses a table to determine what the interest is. The premium you must pay for a given amount of life insurance depends upon your age. The insurance agent looks at a table to tell you what the premium is. The tables you have seen and made in this chapter show dependence; when one of a pair of related numbers is given, you can read the other.

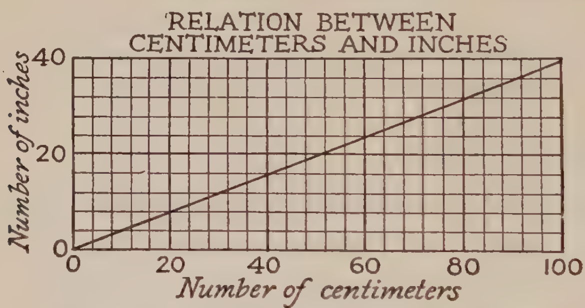
Graphs show quite vividly how one quantity depends upon another, how one changes as the other changes. They picture the table or the formula.

Comparison of Tables, Graphs, and Formulas

We have just said that a bank clerk will use a table to find the interest on a sum of money at a given rate for any required time. Why would he prefer a table to using the formula, $i = prt$? Because in the table the computation has already been done accurately from the formula, so he saves time when he has many computations.

Could the bank clerk use a graph to determine the interest? Yes, but for his purpose the table can be read as quickly and more accurately than a small graph.

Suppose you were working in the office of an industrial plant where it was necessary for you very often to find the number of centimeters in a given number of inches. You know that 1 in. = 2.54 cm. The formula for converting inches to centimeters is $c = 2.54\,i$. You could make each computation accurately, using this formula. If you needed to make some conversions over and over again, you could make a table from the formula for all values you were using. Of course, it would



be wise to check your results. A graph like that shown here, only drawn to a larger scale, would help you to discover any errors at a glance.

If a man had to find the areas of many circles, would he prefer a table, a graph, or a formula? If the table was complete enough, that would be best. The chances are that he might have to find the area of some circles whose radii were not given in the table. If his work did not have to be too accurate, a graph drawn on a large scale would be valuable. The formula is fundamentally the most useful.

In general, formulas are more fundamental than graphs or tables (even though in special cases the latter may be more useful) because (1) for each value of one variable we can compute the value of the other as accurately as we please, (2) we can memorize the formula and use it whenever and wherever we please, (3) by studying the formula we see more accurately how one variable changes with the other. The graph, of course, helps in picturing (3) and in checking computations.

Algebra as Generalized Arithmetic[1]

1. A room is l ft. long and w ft. wide. What is the perimeter in feet? in inches? in yards? What is the area in square feet? in square yards? in square inches?

2. How do you express 3 times the sum of a certain number n and 5?

3. If n represents a certain number of days, what will represent the following?

(*a*) Five times as many days
(*b*) Five more days
(*c*) Three more than twice as many days
(*d*) Six less than 3 times as many days
(*e*) What is the number of inches in n feet? in n feet and p inches? in r yards, s feet, and t inches?

4. If a salesman drives m miles, how long will it take him at the rate of r miles an hour?

5. How much farther will a man go who travels H hours at the rate of R miles an hour than a man who travels h hours at the rate of r miles an hour?

6. A girl is 4 years older than her brother. If the boy is y years old, how old is the girl?

7. A boy is 25 years younger than his mother. The mother is b years old. How old is the boy?

[1] To the Teacher. See Note 3 on page 458.

8. A man has 5 more than twice as many cows as sheep. If he has n sheep, how many cows has he?

9. In a certain triangle, one side is 3 ft. longer than a second side, and the third side is 5 ft. longer than the second side. If the second side is s, what is the perimeter of the triangle?

10. Express in terms of w the area A of a rectangle whose length is 4 ft. more than its width w.

11. Express in terms of w the area A of a rectangle whose length is twice its width w.

12. What is the total cost in dollars of 200 drinks at a soda fountain, n of them at 5 cents and the rest at 10 cents?

13. If A receives n dollars, B \$100 less than that, and C \$500 less than the combined amounts of A and B, how much do B and C receive in terms of n?

14. The width of a rectangle is w ft. Its length contains as many yards as the width contains feet. Express the length in feet. What is the perimeter in feet?

Problem Solving

1. During one afternoon a clerk at a soda fountain sold 200 drinks, some at 5 cents and some at 10 cents each, for which he received \$16. Find the number of each kind.

2. Divide \$3700 among A, B, and C, so that B will receive \$100 less than A, and C \$500 less than the combined amount of A and B.

3. The length of a rectangle contains as many yards as its width does feet. The perimeter is 32 yd. Find the number of feet in each dimension.

4. One can will hold as many quarts as another will hold pints. The combined capacity of both is 18 gallons. Find the capacity of each can in pints.

5. In a collection of bottles there are 5 times as many quart bottles as pint bottles and the combined capacity of all the bottles is 11 gallons. How many bottles of each kind are there?

6. A cab driver received twice as many quarters as half dollars, and 3 times as many dimes as half dollars; in all he received $13. How many coins of each kind did he receive?

7. A is 10 years older than C, and B is 6 years younger than C. The sum of their ages 6 years ago was 40 years. How old is each now?

---

8. A freight train left Kansas City for St. Louis at the rate of 12 miles an hour at the same time that a passenger train running 45 miles an hour left St. Louis for Kansas City. How long will it be before they meet, the distance between the two cities being 285 miles?

9. Two hours after a messenger, traveling at the rate of 10 miles an hour, had left a camp, it was decided to cancel his message. How fast did a second messenger have to travel in order to overtake him in 8 hours?

Chapter Summary

Graphs picture numbers and number relationships far more vividly than does reading matter discussing these numbers or do tables of the numbers themselves. The formula is basic in showing number relationships, but tables and graphs are often more useful.

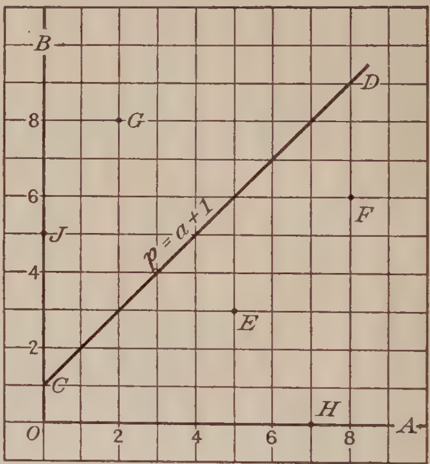
Three types of graphs are discussed in this chapter: pictograms, bar graphs, and line graphs, with special attention being given to algebraic line graphs. *Pictograms* not only picture numbers comparatively but the symbols used to do this are chosen to add meaning to the numbers. Both *bar graphs* and *line graphs* represent numbers as lengths of lines. Once a unit of length is chosen, any number can be represented. *Bar graphs* picture numbers as lengths of lines by using bars of varying lengths or heights.

The *line graphs* in this chapter picture formulas with two variables. They are made by *plotting points* to represent pairs of numbers that satisfy the formula. These numbers are represented by distances from the *horizontal* and *vertical axes.* The line through the points is the *graph of the formula.* It may be straight or curved, depending upon the formula.

From your study of this chapter, you should know the meaning of each of the following technical terms:

Pictogram	Horizontal axis
Bar graph	Vertical axis
Line graph	Origin
Plotting points	Graph of a formula

Chapter Review



1. In the chart at the right, what are OA, OB, and O called? What is CD?

2. What number-pairs are represented by E, F, G, H, and J?

3. On a piece of graph paper locate the points which represent (7, 5), (0, 0), (0, 3), and (4, 0).

4. Draw the graph of $A = 3b + 2$, using the values $b = 3$, 6, and 9. From this graph find the value of A when $b = 5$ and the value of b when $A = 14$. Explain how you did this.

5. How long will it take an airplane to fly n miles at the rate of $r - 10$ miles an hour?

6. The top of a mountain is v feet above sea level. The bottom of a lake in a depressed valley is one fifth as far below sea level as the mountain peak is above it. Express algebraically the difference in altitude of the mountain peak and the lake bottom.

7. The proprietor of a stationery store sold during one day 100 writing tablets at 5 cents and 10 cents each and received 7 dollars. How many tablets did he sell at each price?

CUMULATIVE REVIEW

1. In the equation $3(a + b) = 3a + 3b$, do a and b mean *any numbers* or *some numbers?*

2. In the equation $3a - 5 = 8$, does the a mean *any number* or *some number?*

3. The statement $3(a + b) = 3a + 3b$ means the following: If I add any two numbers and then multiply the sum by 3, the result is the same as if I multiply the first number by 3, then multiply the second number by 3, and then add the two products. What does the statement $3(a - b) = 3a - 3b$ mean? Check the truth of these statements by using $a = 5$ and $b = 2$.

4. What do the following statements mean? (Explain as is done in Ex. 3 above.)

(*a*) $3a + 2a = 5a$

(*b*) $(a + b) + c = a + (b + c)$

(*c*) $a - (b - c) = a - b + c$

(*d*) $(a + b)(a - b) = a^2 - b^2$

(*e*) $\frac{a}{3} + \frac{b}{3} = \frac{a + b}{3}$

(*f*) $\frac{ab}{2} = \frac{a}{2}(b) = a\left(\frac{b}{2}\right)$

(*g*) $3(ab) = (3a)b = (3b)a$

(*h*) $(a + b)^2 = a^2 + 2ab + b^2$

5. Check the statement in Ex. 4 *c* above by using at least one set of values for a, b, and c.

6. Write as equations:

(*a*) 4 more than twice a number is equal to 3 times the number, and

(*b*) 6 less than 3 times a number is equal to the number.

In these two equations is the number *any number* or *some number?*

7. Combine similar terms:

(*a*) $4a + 2b + 3 - 2a + b$
(*b*) $3x + 2 - x + 7 + 5y - 3y - 9$
(*c*) $5m + 2 - 2m + 6m + 9$
(*d*) $x^2 + 2xy - x^2 - xy$
(*e*) $3a + 2.5a + .7a$
(*f*) $\frac{1}{4}m + \frac{2}{3}m + \frac{5}{6}m$
(*g*) $17b - 16.5b + 2.9b$
(*h*) $9\frac{3}{4}y - 7\frac{1}{8}y + 2\frac{1}{2}y$

8. If $x = 5$ and $y = 2$, what is the value of each of the following expressions?

(*a*) $x^2 + 2y$
(*b*) $5x - 3y$
(*c*) $3(x - y)$
(*d*) $(x + y)(x - y)$
(*e*) $\frac{2x + 3y}{3}$

9. If $a = 7$ amd $b = 3$, what is the value of each of the following expressions?

(*a*) $a^2 - 2ab + b^2$
(*b*) $(2a + 3b)(2a - 3b)$
(*c*) $2a + 3b(2a - 3b)$
(*d*) $\frac{3a + 2b}{(a - b)^2}$

10. Find the value of each of the following expressions when $a = 2\frac{1}{2}$ and $b = 3\frac{1}{3}$:

(*a*) $4ab$
(*b*) $3a + 2b$
(*c*) $2a^2 - b^2$
(*d*) $(a + b)^2$
(*e*) $a(2a - b)$
(*f*) $5a - (2a + b)$

11. Multiply as indicated and combine like terms when possible:

(*a*) $7(x + y)$
(*b*) $9(m - n)$
(*c*) $3(7b)$
(*d*) $(7x)(9y)$
(*e*) $(4)(5p)(0)$
(*f*) $7(4n + 2)$
(*g*) $9(3a) + 4(a + 7)$
(*h*) $x(x + y)$
(*i*) $8(2b + 3) + 2(5b + 7)$

12. Divide as indicated:

(*a*) $\frac{6a}{3}$ (*c*) $\frac{9n^2}{3n}$ (*e*) $\frac{7a + 42}{7}$

(*b*) $\frac{6a}{a}$ (*d*) $\frac{18ab}{3b}$ (*f*) $\frac{a^2 + 2ab}{a}$

13. Solve and check the following equations:

(*a*) $a - 7 = 15$

(*b*) $7a = 9$

(*c*) $\frac{a}{7} = 2$

(*d*) $8a + 3a = 33$

(*e*) $a + 2 = 5$

(*f*) $3a - 5 = 9$

(*g*) $.6y = 3$

(*h*) $10 = n - 5$

(*i*) $40 = 10x$

(*j*) $1 = \frac{b}{3}$

(*k*) $1.2a = 12$

(*l*) $x - 0.5 = 3$

(*m*) $18y + 12 - 11y + 4 = 23$

(*n*) $2(3n + 4) = 11$

(*o*) $3(2n + 5) + 5 = 35$

(*p*) $3(2n - 5) - 5 = 35$

14. Write formulas for the following:

(*a*) Perimeter of a rectangle

(*b*) Area of a rectangle

(*c*) Circumference of a circle

(*d*) Area of a circle

(*e*) Area of a triangle

(*f*) Volume of a rectangular solid

(*g*) Volume of a cylinder

15. The interest I can collect on a 100-dollar bond in 1 year depends upon the _?_.

16. The area of a circle depends upon the length of the _?_ of the circle.

17. The formula $s = 16t^2$ shows the number of feet, s, an object will fall in t seconds, due to the force of gravity. How far will it fall in 3 seconds?

18. How many of the following factors would you divide by 5 in order to divide the product by 5? $7 \times 10 \times 15 \times 4$.

19. What is the value of $\frac{12 \times 14}{2}$? Show three ways of getting the answer.

20. The formula for the volume of a cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height. What is the volume of a cone whose radius is 6 in. and whose height is 14 in.? (Use $\pi = \frac{22}{7}$.)

21. The area of a triangle is 18 sq. ft. and the base is 7 ft. What is the height?

22. If the radius of a circle is doubled, what is the effect upon the circumference? the area?

23. If the edge of a cube is multiplied by 3, what is the effect upon the volume?

24. The outside dimensions of a rectangular box are l, w, and h. The inside dimensions are a, b, and c. What is the volume of the wood of which the box is made?

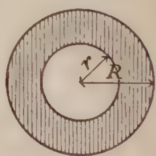
25. Write a formula for the cost (c) in dollars of n articles of the same kind, if one article costs p cents.

26. The formula for the total area of the surface of a cylinder, including the two bases and the curved surface, is $2\pi r(r + h)$.

Find the area to the nearest tenth, when $r = 3.5$, $h = 2.3$, and $\pi = \frac{22}{7}$.

27. Is x^2 ever equal to x, ever greater than x, or ever smaller than x? Illustrate.

28. $A = \pi(R + r)(R - r)$ is the formula for the area of a ring made by circles, as in the figure. Find to the nearest tenth the area of the ring if $R = 3.2$ in. and $r = 2.5$ in. (Use $\pi = \frac{22}{7}$.)



29. Two cans of fruit juice have the same diameter, but the first is twice as tall as the second. The first will hold ___?___ times as much as the second. ($V = \pi r^2 h$.)

30. Two cans of fruit juice have the same height, but the diameter of the first is twice the diameter of the second. The first will hold ___?___ times as much as the second.

31. Write a formula for the area (A) of a circle in terms of its diameter (d). (Make use of the fact that $d = 2r$.)

Douglas Aircraft

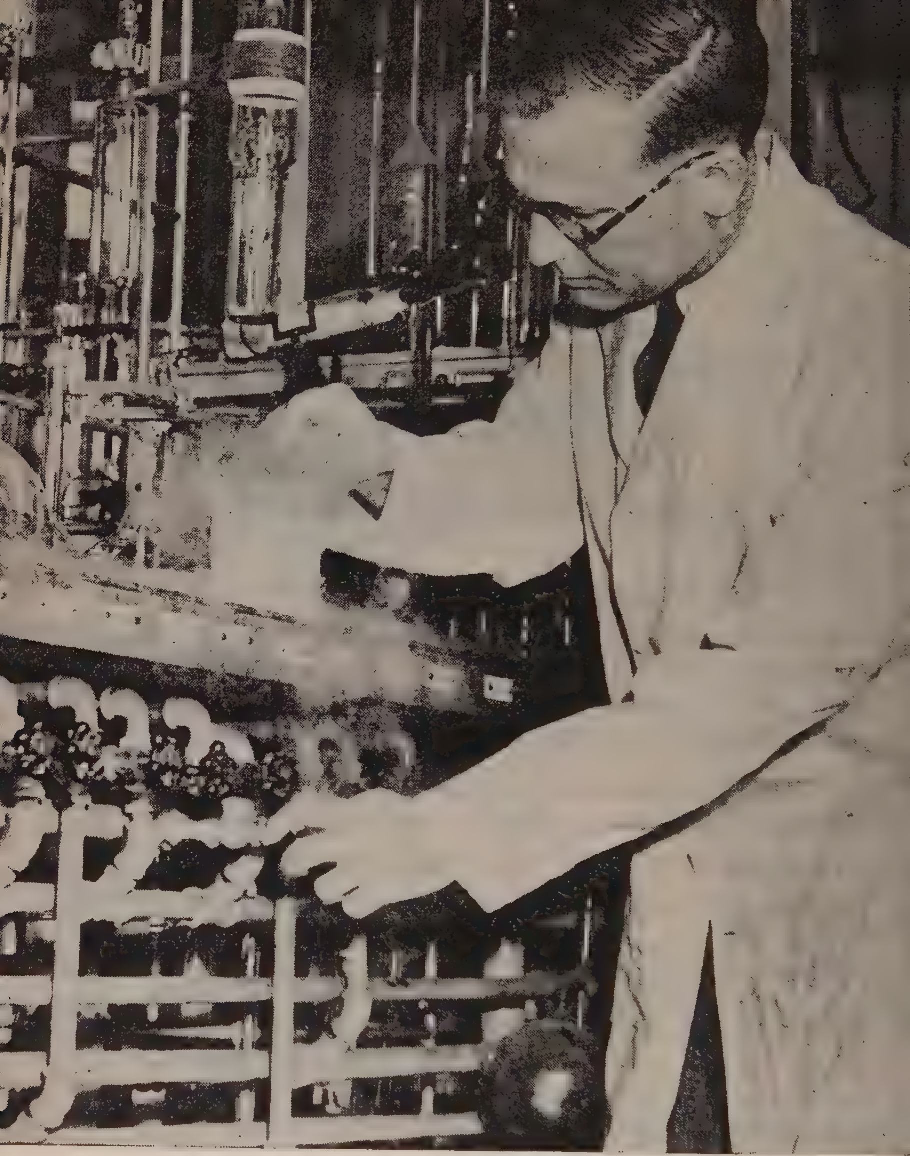
Using Formulas in Aviation

1. A propeller blade on a plane has a length of 4 feet. How far will the tip of the blade travel in making a complete revolution? How many feet per minute will the tip travel if the propeller makes 700 revolutions per minute?

2. The distance a plane can travel from its home field and return without running out of fuel is called its *radius of action.* A formula for finding the radius of action is $R = \left(\frac{ab}{a+b}\right)t$, where R denotes the radius of action in miles, a the outward speed in miles per hour, b the returning speed, and t the time of the journey in hours.

A pilot knows that on the outward trip of an observation flight, his plane will make 300 miles an hour because of a tail wind; but returning, his speed will be 250 miles an hour on account of the head wind. He knows also that his plane carries fuel for 6 hours of flying. He will, however, keep one hour's supply of fuel for emergency, and so plan a 5-hour flight. Use the formula to compute his radius of action for that flight.

3. What would be the radius of action for a flight of 6 hours if the outward speed is 180 miles per hour and the returning speed is 240 miles per hour?



Brown Brothers

A scientist in the General Electric Research Laboratory performing an experiment with liquid air. Air is composed of a mixture of gases, the most abundant of which are nitrogen, oxygen, and argon. When liquid air is allowed to evaporate the nitrogen changes to a gas ("boils off") at $-195.5°$ C., the argon at $-186°$ C., and the oxygen at $-182.5°$ C. The scientist who works with liquid air deals with negative numbers.

Chapter V

SIGNED NUMBERS[1]

An advertisement in a recent magazine began: "Up — Up — Up, 40,000 feet and beyond! Where temperatures drop to $-60°$ and below." The financial pages of the newspapers abound with numbers like $-\frac{1}{4}$, $+\frac{5}{8}$, $-\frac{7}{8}$. It is assumed that the person who reads these numbers with plus or minus signs before them knows what they mean. Do *you?*

Numbers with a plus or minus sign before them are called ***signed numbers***. They are of two kinds, ***positive*** and ***negative numbers***. A plus sign indicates a positive number; a minus sign indicates a negative number. Positive and negative numbers are opposites and, as you will see, are convenient to describe many of the situations that arise in life. You have often used these numbers in work or in play.

Meaning and Use of Signed Numbers

(1) The pupils of an eighth-grade class were asked what answer they would get if they subtracted 9 from 5. They said it could not be done. They were then asked, "If, in playing a game, you had 5 points and then lost 9 points, what would your score be?" They answered, "Four points in the hole." That was their way of stating a number which was *opposite* to "Four points to the good." Were they correct in saying that 9 cannot be subtracted from 5?

(2) Businessmen need two kinds of numbers. If a man, at the end of a business year, finds that he has lost \$1000 instead of making a gain, he may write the \$1000 in his books in red

[1] To the Teacher. See Note 12 on page 460.

ink and say, "I am \$1000 in the red." That is his way of stating a number which is *opposite* to \$1000 in the black or "\$1000 ahead of the game."

(3) On temperature scales there are numbers opposite in kind separated by the zero point. If the thermometer reading is now zero and the temperature rises 4 degrees during the next hour, what will the reading be? Suppose that the temperature drops 6 degrees during the second hour, what will the reading be? One way to state this reading would be *2° below zero.* That states a number *opposite* to 2° above zero.

(4) Mathematicians use plus and minus signs before numbers opposite in nature. For "four to the good" they write $+4$ (read "plus four"); for "four in the hole," -4 (read "minus four"). For "\$1000 ahead of the game" they write $+\$1000$; for "\$1000 in the red," $-\$1000$. How would they write "2° above 0" and "2° below 0"?

A positive number is indicated by a plus sign. A negative number is preceded by a minus sign. The minus sign is placed before a negative number even when it stands alone. If no sign precedes a number, it is a positive one. You should not confuse the signs attached to signed numbers with the addition and subtraction signs, which are signs of operation to indicate when additions and subtractions are to be performed. You should have clearly in mind these two distinct uses of plus and minus signs.

Exercises

1. In the following examples, state the quantity which is opposite to the one given:

(*a*) 10° above 0
(*b*) \$150 profit
(*c*) 15 miles south
(*d*) 8 feet up
(*e*) 10 steps backward
(*f*) 7 miles west
(*g*) 8 pounds overweight
(*h*) 10 points to the good
(*i*) \$10 deposit
(*j*) \$15 in debt
(*k*) 500 feet above sea level
(*l*) 5 to the left
(*m*) \$1.50 spent
(*n*) 8 yards gain
(*o*) 55 B.C.
(*p*) $+72$

2. If + 3 represents a number of pounds overweight, what would − 3 represent?

3. If + 10 represents a number of miles north, what would − 10 represent?

4. If + 5 represents 5 to the right, what will − 5 represent?

5. If + 100 represents a number of dollars gained, what would − 100 represent?

6. If + 15 represents a number of degrees' rise in temperature, what would − 15 represent?

7. If you say that Mr. Driggs's financial situation is − $500, what do you mean? How much will it have to improve before he will be $500 "to the good"?

8. If we speak of noon today as zero, what number represents 9 o'clock this morning? 4 o'clock this afternoon? (Consider the time before the zero hour as negative.)

9. If we call midnight of last night zero, what number represents 6 o'clock last evening? what number represents 10 o'clock this morning?

10. If one boy walks + 3 miles and another walks − 3 miles, do they walk equal distances?

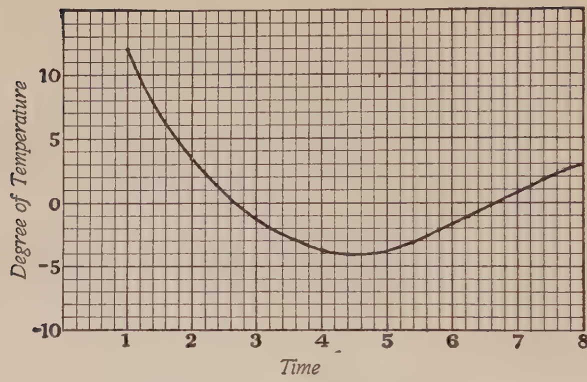
11. An autoist drove 75 miles due east from his starting point. What reading on this distance-direction scale below represents his stopping place?

Readings to the east (right) of the zero point are usually considered *positive;* those to the west (left) are then *negative.*

12. If the autoist drove 50 miles east and then 75 miles west, how far and in what direction would he be from his starting point? Would you represent your answer by a positive or a negative number?

13. Using the scale above, explain the meaning of —
(*a*) + 100, (*b*) − 25, (*c*) + 200, (*d*) − 100.

A TEMPERATURE GRAPH



14. From the graph above tell what the temperature was at 1 o'clock, 2 o'clock, and 3 o'clock.

15. What was the reading at 4 o'clock? at 5 o'clock? at 6 o'clock? at 7 o'clock? At what hour was the reading 5°? When was it zero? How many times was it zero?

16. Making use of the scale below, determine the final position of a boy who, starting at 0, walked 20 blocks east, then 5 blocks west, then 15 blocks east, and then 40 blocks west. How far was he from the starting point and in what direction? Would you represent the distance by a positive number or a negative number?

17. Again using a scale, determine the final position of a girl who, starting at 0, walked 16 blocks west, then 9 blocks east, then 18 blocks west, then 7 blocks east, and then 18 blocks east.

18. Eight children were playing a game called "beanbag." Each had two trials. If they had known about positive and negative numbers, they could have written their scores as below. Find the final score of each.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
-4	$+3$	$+3$	$+7$	$+5$	-7	$+8$	-7
$+3$	$+4$	-5	-3	$+4$	-3	-3	$+3$

19. Grace's scores in a game were $+10$, $+16$, -8, and $+5$. Find her total score.

20. Tom's scores in a game were -6, -8, $+5$, and -7. What was his final score?

21. A tail wind increases the speed of an airplane and a head wind decreases the speed an amount equal to the velocity of the wind. Using appropriate signs, indicate the effect on the speed of a plane of a head wind of 40 m.p.h. (miles per hour) and of a tail wind of 60 m.p.h.

22. At 6 o'clock in the evening the reading of a thermometer was 0. If the temperature drops 2 degrees an hour until midnight, what signed number will represent the temperature at 9 P.M.? at 10 P.M.? at 11.30 P.M.? at midnight?

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23. What must you add to 2 to make 6?

24. In each case below, tell what you must add to the lower number to get the upper number.

7	7	-7	-7	4	4	-4	-4	4	0	-4	0
4	-4	-4	4	7	-7	-7	7	0	4	0	-4

Examples of the Use of Signed Numbers

Below are replies given by five pupils in answer to the problem: Write an example of positive and negative numbers which you yourself have used or which you have seen or heard other people use.

Are the statements correct? Can you add any examples of your own?

(1) If we think of any direction, say north, as positive, then we can speak of the opposite direction, south, as negative. In the same way 5 miles east may be thought of as $+5$, whereas 3 miles west is -3.

(2) In playing games, I may lose until my score is -10, which means I have to get 10 points before I am back to zero. Or I may win 10 points without losing anything, in which case my score is $+10$.

(3) In my general science textbook and in *Popular Mechanics* I read about a weight or a pressure acting downward. A balloon inflated with a gas lighter than air would have an upward pull. The upward force may be thought of as negative, but a downward force is thought of as positive.

(4) In geography, when we locate places north or south of the equator, we use North and South Latitude. 5° North Latitude could be written as + 5°, and 5° South Latitude would then be − 5°. The same plan may be used in measuring longitude east and west from a selected meridian.

(5) People deposit money in a bank and draw out money by checks. A $50 deposit is thought of as a + $50, and the writing of a check for $50 to be charged against the account as a − $50, in balancing a bank account.

(6) This way of writing numbers which are opposite in nature is now frequent in newspapers. Bring to class clippings in which these numbers are used and tell what they mean.

Exercises

1. A blimp patrolling a coast line headed out to sea 34 miles, turned shoreward for 27 miles, went seaward for 30 miles, and then turned back toward the shore for 18 miles. At the end of the period how far was the blimp from shore?

2. A plane has a speed of n m.p.h. in still air. Write algebraically its speed when flying with a tail wind of 50 m.p.h.; with a head wind of 20 m.p.h.

Goodyear Rubber Company

3. The highest mountain in the world is Mount Everest with an altitude of 29,141 feet. The average depth of the sea is 12,450 feet. The deepest place found in the sea is 35,400 feet. Write these figures with appropriate signs.

How would you indicate the elevation of the surface of the sea?

4. The latitude of Savannah, Georgia, is approximately 32° North and of Rio de Janeiro 23° South. What signs would you affix to each of these numbers? How far apart are the two places in degrees?

5. Plato was born -427. Charlemagne was born $+768$. How far apart were the two events in time?

Chapter Summary

In arithmetic the numbers you study are *non-directed.* When we use them, we have no thought of whether they are plus or minus. They tell only *how much* or *how many.*

Signed numbers do more than this. Whenever it is possible to think of two numbers as opposite in nature, the plus or minus sign tells the direction of the number — whether it is on one side of the zero point or the other. Sometimes signed numbers are called *directed numbers.*

As to one fact you should not become confused. The essential idea in signed numbers is that of *oppositeness.* Positive and negative numbers indicate quantities that are opposite in quality. Zero does not necessarily represent nothing, but a point arbitrarily decided upon. A man who walks -5 miles does not walk 5 miles less than nothing, but 5 miles in a direction opposite to that which has been chosen as positive. On a centigrade thermometer 0° does not represent no temperature, but the point at which water freezes. Think of positive and negative numbers as numbers that are opposite in character.

It is important to remember that in algebra plus and minus signs have two distinct uses. When they are used to indicate whether a number is positive or negative, they are called *signs of quality.* When they are used to indicate addition and subtraction, they are called *signs of operation.*

Chapter Review

1. If $+20$ represents one of the unit marks on the scale of a thermometer, how many units is that from the zero mark?

2. If -20 represents one of the unit marks on the scale of a thermometer, how many units is that from the zero mark?

3. Which is a higher reading on a thermometer, $+20°$ or $+5°$? $+10°$ or $-20°$?

4. If a certain stock on the New York Exchange sells for 6 in the morning and $6\frac{3}{4}$ at closing time, would you indicate the change by $+\frac{3}{4}$ or by $-\frac{3}{4}$?

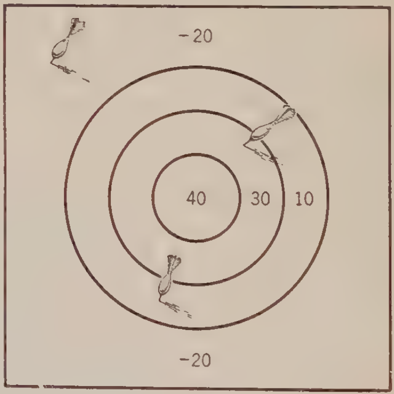
5. If you saw the statement that the lowest point in a certain valley is -200 ft., what would the statement mean to you?

6. The change in the level of the Connecticut River last night was $-.3$ ft. Explain what that means.

7. The following grades were received by six boys in an algebra class:

Pupil. . . .	A	B	C	D	E	F
Grade . . .	98	95	73	78	84	88

What was the average grade? Which pupils were above the average grade? Which below? For each pupil, express by a signed number the number of points above or below the average.



8. Three young people were throwing darts at the target shown at the right. Their scores on three trials each are given below. What are their final scores?

A.	B.	C.
40	30	-20
30	-20	10
10	10	-20

Maintaining Skills

(Fractions)

1. Write each of the following as a whole or a mixed number. Reduce your answer to lowest terms.

(*a*) $\frac{15}{2}$ (*c*) $\frac{43}{5}$ (*e*) $\frac{82}{15}$

(*b*) $\frac{29}{3}$ (*d*) $\frac{36}{12}$ (*f*) $\frac{304}{16}$

2. Change these mixed numbers to improper fractions:

(*a*) $5\frac{1}{2} = \frac{}{2}$ (*c*) $8\frac{4}{5} = \frac{}{5}$ (*e*) $32\frac{7}{16} = \frac{}{16}$

(*b*) $7\frac{3}{4} = \frac{}{4}$ (*d*) $9\frac{7}{12} = \frac{}{12}$ (*f*) $205\frac{5}{12} = \frac{}{12}$

3. Change to equivalent fractions:

(*a*) $\frac{3}{4} = \frac{?}{8} = \frac{?}{12}$ (*c*) $\frac{5}{8} = \frac{?}{16} = \frac{?}{48}$

(*b*) $\frac{2}{3} = \frac{?}{9} = \frac{?}{24}$ (*d*) $\frac{8}{12} = \frac{?}{24} = \frac{?}{36}$

4. Add as indicated:

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)
$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$4\frac{1}{2}$	$6\frac{2}{3}$
$\frac{3}{4}$	$\frac{2}{5}$	$\frac{7}{12}$	$\frac{3}{4}$	$9\frac{3}{4}$	$3\frac{3}{4}$
$\frac{5}{6}$	$\frac{7}{10}$	$\frac{5}{6}$	$\frac{7}{8}$	$6\frac{5}{8}$	$5\frac{1}{2}$

(*g*) $2\frac{3}{5} + 6\frac{1}{2} + 7\frac{5}{6}$ (*h*) $8\frac{3}{4} + 2\frac{1}{5} + 3\frac{7}{10}$

5. Subtract the lower number (subtrahend) from the upper number (minuend):

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)
$\frac{5}{8}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{9}{10}$	$8\frac{3}{5}$	$8\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{6}$	$4\frac{1}{2}$	$3\frac{3}{4}$

(*g*) $7\frac{5}{6} - 3\frac{3}{4}$ (*h*) $2\frac{3}{7} - 1\frac{1}{2}$

6. Multiply as indicated:

(*a*) $9 \times \frac{2}{3}$ (*c*) $\frac{1}{2} \times \frac{1}{3}$ (*e*) $7\frac{3}{5} \times 9$

(*b*) $\frac{3}{4} \times 12$ (*d*) $\frac{2}{3} \times \frac{3}{4}$ (*f*) $3\frac{1}{2} \times 7\frac{1}{2}$

7. Divide as indicated:

(*a*) $\frac{5}{4} \div \frac{1}{8}$ (*c*) $8\frac{2}{3} \div \frac{2}{3}$ (*e*) $6 \div \frac{1}{2}$

(*b*) $\frac{2}{3} \div \frac{1}{2}$ (*d*) $5\frac{1}{2} \div 6$ (*f*) $4\frac{2}{5} \div 3\frac{1}{2}$

A Note on Signed Numbers

The first mathematician of the western world to mention negative numbers was Diophantus, who lived in Alexandria in the third century A.D. He called a positive number a "fourth-coming" and a negative number a "wanting." Yet he did not understand the real meaning of a negative number as one opposite in character to a positive number. He thought that an equation like $x + 16 = 4$ is absurd, since it would give $x = -12$.

During the Middle Ages only a few ideas were added to the theory of negative numbers. But in 1637 René Descartes, a French mathematician, published a famous book on geometry, in which he employed graphs of equations and used the idea of opposite directions. Then the meaning of negative numbers was finally made clear.

René Descartes, 1596–1650, who contributed much of importance to the science of mathematics. He originated the idea of graphing equations and added especially to our knowledge of negative numbers.

Chapter VI

ADDITION AND SUBTRACTION OF ALGEBRAIC NUMBERS[1]

The first thing you learned about algebra was its symbolism. When you mastered this, you could get the answers to certain kinds of problems by substituting numbers in algebraic formulas. For this kind of work familiarity with algebraic symbolism and a knowledge of arithmetic were all you needed. But to solve certain kinds of equations you soon had to learn how to change the form of an algebraic expression.

For example, in solving the equation $2a + 3a = 10$, you had to change $2a + 3a$ to $5a$ by combining terms. This is an example of adding literal numbers (pages 9 and 11). Further along in your work, in order to solve an equation like $3(n + 2) = 17$, you had to change $3(n + 2)$ to $3n + 6$. This is an example of multiplying literal numbers (pages 47–48). You learned other operations also.

As you progress in algebra you will have to solve more complex equations which require more complex operations than these. To do them you will have to understand and learn the definite rules for addition, subtraction, multiplication, and division of algebraic numbers just as you had to learn how to do these operations in arithmetic. In all algebraic computation, an understanding of signed numbers and the ability to operate with them in connection with literal numbers is of vital importance.

[1] To the Teacher. See Note 13 on page 460.

Enlarging the Meaning of Addition

You know what it means to add 6 and 4. It may seem strange to you to *add* 6 and -4, for you have become accustomed to the minus sign as meaning subtraction. The following illustrations will help you to understand what is meant by adding positive and negative numbers. This is really not new to you because you have already added a few signed numbers in some of the exercises in the preceding chapter.

(1) In the morning mail I receive several checks and several bills. The first envelope contains a check for \$50, the second a bill for \$35, the third a bill for \$20, the fourth a check for \$10, the fifth a check for \$15. What is the *total* effect of all these upon my financial standing? Finding the answer is called *adding* because it is the result of *putting together* several separate items.

(2) Ex. (1) might be written: Add $+50$, -35, -20, $+10$, and $+15$.

(3) I walk 6 ft. east, then 10 ft. west, then 3 ft. east, then 5 ft. east, then 9 ft. west. How far am I from my starting point? Finding the answer is here also called *addition*, since the *total* effect of these various distances is called for.

(4) Ex. (3) could be written: Add $+6$, -10, $+3$, $+5$, and -9.

(5) Here are the scores made by eight persons in six trials each, in a game. Find the final score for each person by adding the scores he made in the six trials.

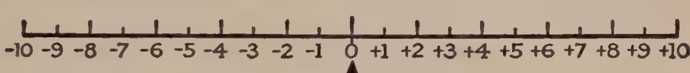
(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)	(*g*)	(*h*)
4	1	7	8	-4	5	-2	5
-3	4	-4	-5	-2	3	-1	-2
-1	3	-2	4	7	-4	-4	3
6	-3	3	-3	1	-3	5	-2
2	3	2	-2	-3	3	3	1
3	-4	6	5	-2	-7	-6	-5

Douglas Aircraft Co.

(6) An airplane rose to a height of 20,000 feet, descended 17,000 feet, and then to get above a cloud bank rose again 2000 feet. Write these figures with appropriate signs. What was the final altitude of the plane?

The Number Scale

You learned in a preceding chapter that a number can be represented by a length of line. We can make use of this principle in representing signed numbers on a line as in this scale.



Number scale

As you see, positive numbers are indicated to the right of the zero mark, and negative numbers (opposites) to the left.

If you begin at 0, the zero mark, and move 3 units to the right, you reach the mark $+3$. The length of line from 0 to $+3$ represents $+3$. If you begin at 0 and move 3 units to the left, you reach the mark -3. The length of line from 0 to -3 represents -3. Similarly for all the other numbers.

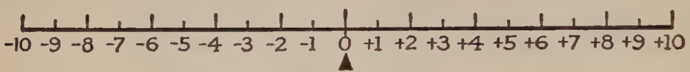
If you proceed from the $+3$ mark to the $+7$ mark, you have gone 4 units *to the right*. The change is $+4$.

If you proceed from $+7$ to $+3$ or from $+3$ to -1, you have in each case moved 4 units *to the left*. The change is -4.

Moving 5 units to the right from any point is equivalent to adding $+5$. Moving 5 units to the left from any point is equivalent to adding -5.

Adding Signed Numbers by Use of a Scale

Using the number scale below will help you further in understanding what addition of positive and negative numbers means.



Number scale

(1) Add: + 2 and + 3. Start at 0, go 2 units to the right; then go 3 more units to the right. What is your answer?

(2) Add: + 2 and – 3. Start at 0, go 2 units to the right; then go 3 units to the left.

(3) Add: – 2 and – 3. Start at 0, go 2 units to the left; then go 3 more units to the left.

(4) Add: – 2 and + 3. Start at 0, go 2 units to the left; then go 3 units to the right.

Exercises

1. Using the number scale, add the following pairs of numbers. (When no sign is written before a number, the plus sign is understood.)

(*a*) + 8 + 5	(*e*) 8 5	(*i*) – 5 + 8	(*m*) 0 + 7
(*b*) + 8 – 5	(*f*) 8 – 5	(*j*) 5 – 8	(*n*) 7 0
(*c*) – 8 + 5	(*g*) – 8 5	(*k*) – 5 – 8	(*o*) 0 – 7
(*d*) – 8 – 5	(*h*) + 5 + 8	(*l*) 5 + 8	(*p*) – 7 0

2. Add these numbers: + 8, – 5, – 3, + 2.

3. An elevator went up to the 9th floor, came down 4 floors, went up 2 floors, and descended 5 floors. How far was it then from the ground floor? Solve the problem, using the number scale.

Rules for Addition of Signed Numbers

You can now discover a rule for adding signed numbers.

(1) Choose the correct word in each of the following. Use the number scale to get the sum in each case.

(*a*) If I add + 8 and + 6, the sum is (positive, negative).
(*b*) If I add − 8 and − 6, the sum is (positive, negative).
(*c*) If I add − 8 and + 6, the sum is (positive, negative).
(*d*) If I add + 8 and − 6, the sum is (positive, negative).

Exs. (2) and (3) below will lead you to a rule for adding positive numbers.

(2) Add, using the number scale:

+ 3	+ 8	+ 9	+ 8	7	16	11	10
+ 7	+ 5	+ 12	+ 6	16	9	12	15

(3) In Ex. (2) the addends are all positive and the sums are all _?_. The sum of two (or more) positive numbers is always a _?_ number.

Exs. (4) and (5) will lead you to a rule for adding negative numbers.

(4) Add, using the number scale:

− 3	− 8	− 9	− 8	− 7	− 16	− 11	− 10
− 7	− 5	− 12	− 6	− 16	− 9	− 12	− 15

(5) In Ex. (4) the addends are all negative and the sums are all _?_. The sum of two (or more) negative numbers is always a _?_ number.

(6) Can you make a rule for adding positive numbers? for adding negative numbers? Try it.

The numerical value of a positive or a negative number, without regard to its sign, is called its ***absolute value.*** For example, + 3 and − 3 are both 3 units from the zero mark on the number scale. The distance in both cases is 3, an absolute value without regard to the sign. The signs tell in which direction + 3 and − 3 are. The absolute value of both is 3.

To add two (or more) positive numbers, find the sum of their absolute values and place a plus sign before the result.

To add two (or more) negative numbers, find the sum of their absolute values and place a minus sign before the result.

EXAMPLE 1. To add -9 and -7, find the sum of the absolute values, 9 and 7, and then put a minus sign before the result. Answer: -16.

EXAMPLE 2. The sum of $+9$ and $+7$ is $+16$. Explain.

(7) Add the eight pairs of numbers in Exs. (2) and (4) on the preceding page, using the rules.

Have your classmates make up examples in which the addends are either all positive or all negative and see how quickly you can give the answers.

Exs. (8) to (10) will lead you to a rule for adding a positive and a negative number.

(8) Add, using the number scale (page 146):

$+8$	-8	-5	$+5$	-9	$+9$	-8	$+8$
-5	$+5$	$+8$	-8	$+5$	-5	$+8$	-8

(9) In each of the examples in Ex. (8), one addend is positive and the other is negative. The sums are sometimes __?__, sometimes __?__, and sometimes zero.

(10) In Ex. (8), do you get the answers by finding the sum of the absolute values or the difference between the absolute values? Can you tell whether the answer will be positive or negative just by looking at the addends?

(11) Can you make a rule for adding a positive and a negative number without using the number scale? Try it. Then check your rule by the following:

To add a positive and a negative number, find the difference between their absolute values. Place before the result the sign of the number which has the greater absolute value.

EXAMPLE 1. To add -9 and $+7$, find the difference between 9 and 7 and place a minus sign (the sign of the number of greater absolute value) before the result. Answer: -2.

EXAMPLE 2. The sum of $+9$ and -7 is $+2$. Explain.

Exercises

Add, using the rules on page 148. Check by thinking of the number scale.

1.	-3	$+6$
2.	$+3$	$+6$
3.	-3	-6
4.	$+3$	-6
5.	-4	$+4$
6.	-9	6
7.	2	-5
8.	-7	-2
9.	-4	-9
10.	7	5
11.	-3	$+7$
12.	-10	-3
13.	$+6$	-2
14.	-9	-6
15.	$+8$	-3
16.	6	2
17.	-10	4
18.	-8	$+3$
19.	9	0
20.	0	9
21.	-9	8
22.	$+7$	-3
23.	$+8$	-7
24.	-3	-7
25.	9	6
26.	-6	3
27.	5	-6
28.	-3	7
29.	-9	0
30.	0	-9
31.	8.6	-3.7
32.	-4.5	2.4
33.	8.0	-3.2
34.	-6.0	-7.2
35.	8.2	6.4
36.	1.3	-0.7
37.	2.4	-4.5
38.	-3.7	8.6
39.	-4.0	0.7
40.	-1.6	-6.0

When adding more than two positive and negative numbers, it is usually easier to add first all the positive numbers, then add all the negative numbers, and finally add the resulting positive and negative numbers.

Add the numbers in the following examples:

41.	42.	43.	44.	45.	46.
$+5$	5	-8	7	-8	-12
-8	-7	9	4	-6	-29
$+9$	-8	7	-9	$+9$	$+37$
-2	-3	-5	8	-8	-18

47.	48.	49.	50.
$6n$	$8n$	$-8n$	$-8n$
$-4n$	$-5n$	$-5n$	$5n$

51.	52.	53.	54.	55.	56.
$+4d$	$-8d$	$+13f$	$-27t$	$-58p$	$+9.2p$
$+9d$	$-9d$	$-9f$	$-19t$	$+97p$	$-4.6p$
$-5d$	$+5d$	$+17f$	$+18t$	$-19p$	$-0.8p$

In the following examples the addends are arranged horizontally instead of vertically. Addition is indicated by the plus signs between the parentheses.

57. $(+2)+(+3)=$? **59.** $(-3)+(+2)=$?

58. $(-2)+(-3)=$? **60.** $(+2)+(-3)=$?

61. $(6)+(-7)+(-4)=$?

62. $(-8)+(+9)+(+5)=$?

63. $(+9)+(-6)+(-7)=$?

64. $(-3)+(-4)+(-8)=$?

65. What is the value of $a+b$ when $a=-5$ and $b=-7$? Think $(-5)+(-7)=$?.

66. What is the value of $m+n$ when $m=-5$ and $n=+2$?

Some Algebraic Terms

An algebraic expression that contains only one term is called a *monomial.* (What does the *mono* in "monoplane" mean?)

Examples of monomials are $4x$, $5y^3$, $3xy$, -7, and n.

An algebraic expression containing more than one term is called a *polynomial.* (The *poly* comes from a Greek word meaning "many." You are perhaps familiar with it in the word "polygon.")

Examples of polynomials are $4x - 2$, $5b^2 - 4b + 1$, $a + b$, and $x^2 - 2xy + y^2$.

A polynomial that contains two terms is called a *binomial.* (What does the *bi* in "bicycle" and "biplane" mean?)

Examples of binomials are $a + b$, $2a - 3b$, and $x^2 - 2xy$.

A polynomial of three terms is called a *trinomial.* (Do you know any other words beginning with *tri*, meaning "three"?)

Thus $x^2 - 6x + 8$ and $a + b - c$ are trinomials.

Any factor of a product may be called the *coefficient* of the other factor or factors. In most cases, however, when coefficient is mentioned, it refers to the *numerical coefficient* (page 7).

In the monomial $5xy$, 5, x, and y are the three factors of this indicated product, and 5 is the numerical coefficient of xy. We can, however, say that x is the coefficient of $5y$ or that y is the coefficient of $5x$.

Terms which have the same literal part are called *like terms* (page 11). $2a$, $-7a$, and a are like terms. $2x^2y$ and $-9x^2y$ are like terms. $3ab$ and $3a^2b$ are *unlike terms.*

Exercises

1. Tell which of the following are monomials, which are binomials, and which are trinomials:

(*a*) $a + b + c$
(*b*) $x^2 - 1$
(*c*) $-3xy$
(*d*) $4x^2y^2$
(*e*) $a + b$
(*f*) $a^2 - 2ab + b^2$

2. The expression $2n$ is a _?_ made up of _?_ factors. The _?_ of n is 2.

3. The terms $3x$ and $4x$ are ___?___ terms; $3x^2$ and $3y^2$ are ___?___ terms. The sum of $3x$ and $4x$ is ___?___; the sum of $3x^2$ and $3y^2$ is ___?___.

4. In the expression $2a - 3ac + 5c^2$, how many terms are there? Is it a monomial, binomial, or trinomial? What is the numerical coefficient in the second term? How many factors has the first term? How many factors has the second term?

Addition of Monomials

You already know how to add monomials (page 11). Here you will review the method.

(1) Check the following statements:[1]

$$\begin{aligned} 2(5) + 4(5) &= 6(5) \\ 2(6) + 4(6) &= 6(6) \\ 2(7) + 4(7) &= 6(7) \\ 2(42) + 4(42) &= 6(42) \end{aligned}$$

$$\begin{array}{ccccccc} & & & & & & 5 \\ & & & & & & 5 \\ & & & & 5 & & 5 \\ & & & & 5 & & 5 \\ & 5 & & & 5 & & 5 \\ & 5 & & & 5 & & 5 \\ 2 \times & \overline{5} & + & 4 \times & \overline{5} & = 6 \times & \overline{5} \end{array}$$

Are these statements true because of the particular numbers we have chosen or would the pattern of procedure be the same with any numbers?

These exercises illustrate the fact that 2 times *any number* plus 4 times *that number* equals 6 times *that number;* that is, $2a + 4a = 6a$.

Similarly, $2x^2y + 4x^2y = 6x^2y$ because x^2y is a number just as a is; and $2(a + b) + 4(a + b) = 6(a + b)$ because $a + b$ is a number.

Likewise, $7a + (-3a)$ or $7a - 3a = 4a$.

These illustrations suggest the following rule:

> **To add two monomials which are like terms, find the sum of the numerical coefficients. Then indicate that this sum is to be multiplied by the common literal part.**

Thus, to find the sum of $5p$, $-4p$, $-8p$, and $+3p$, add the coefficients 5, -4, -8, and 3. This gives you -4. Then indicate that this sum of the numerical coefficients is to

[1] TO THE TEACHER. See Note 14 on page 461.

be multiplied by the common literal part, p. This gives -4 times p, or the final sum, $-4p$, as it is indicated in algebra.

Combining the like terms of a polynomial follows the same rule as adding monomials which are like terms. *Only like terms can be combined.* If you did not know the value in cents of a nickel or a dime, you could not find the value of 2 nickels and 5 dimes. All you could say is, "I have 2 nickels + 5 dimes." Similarly, you can find the sum of unlike terms such as $2a$ and $3b$ only by indicating the sum; that is, by writing $2a + 3b$.

To simplify an expression like $3a + 2b - 5a + 7b$, combine the like terms. The result is $-2a + 9b$.

Exercises

Find the sum of the following monomials:

1.	2.	3.	4.	5.
$-2a$	$+7b$	$-2xy$	$-5pq$	$9x^2y$
$-3a$	$-5b$	$+7xy$	$-4pq$	$6x^2y$

6.	7.	8.	9.	10.
$-4m$	$+12x^2$	$+9mn$	$-17ab$	$10x^3$
$-9m$	$-7x^2$	$-16mn$	$+3ab$	$-3x^3$

11.	12.	13.	14.	15.
$+7x$	$-8n$	$+10x$	$-6p$	$+4pq$
$+3x$	$+5n$	$-3x$	$+9p$	$-8pq$
$+6x$	$-2n$	$-6x$	$-5p$	$-6pq$

16.	17.	18.	19.	20.
$-4n$	$+5p^2$	$+7rs^2$	$3m^3$	$2(a+b)$
$+8n$	$-9p^2$	$-4rs^2$	$-4m^3$	$-5(a+b)$
$-7n$	$-3p^2$	$+rs^2$	$-m^3$	$7(a+b)$

Simplify the following polynomials by combining like terms:

21. $6x + 5x - 8x - 4x$
22. $7y + 33y - 14y - 6y$
23. $2r - r + 3r - r$
24. $-8x + 5x - x - 3x$
25. $2a - 3a - 5a + 6a$
26. $2x + 3 + 4x - 7$

27. $-4a+b-2b+c$

28. $2x+2z+3x-2z$

29. $3x-2y-5x-2y$

30. $5a+5b+6b-8a$

31. $x^2-x+3x+y^2$

32. x^2-2-7x^2+3x

33. $3a^2+4a+a^2+2a$

34. $12bc+9ac-3bc$

35. $3x+7x-6+2x-5$

36. $x^2-5x+4-2x-1$

37. $5a^2-2a+2a-5$

38. $2xy-5xy-8+5xy+7$

39. $4ab-ab+2a+a$

40. $x^3+x^2+x+1-3$

41. $3a^2b+2ab^2-7a^2b$

42. $a^3+2a^2-5a^2+7a$

43. $3\frac{1}{4}n-2\frac{1}{2}n-1\frac{1}{4}n$

44. $7.2a+9.1b-5.3a+0.2a$

45. $35{,}000a-a+1000a$

46. $998a+764a+835a+906a$

47. $807b-9b+390b+500b$

Combine like terms:

48. $3.7a-3.2a+0.9a$

49. $6.3a+5a-9.6a$

50. $7.8ac-10ac+2.3ac$

51. $-3.5n^2+n^2-2.5n^2$

52. $\frac{7}{8}ax^2-4ax^2+\frac{9}{8}ax^2$

53. $2\frac{2}{3}x^2yz-5\frac{1}{3}x^2yz-7\frac{1}{6}x^2yz$

54. $8\sqrt{a}-4\sqrt{b}+20\sqrt{a}+9\sqrt{b}$

55. $x+3(a+1)-4(a+1)+7x$

56. $5a^x-3b^y+4b^y-a^x+7a^x+4$

57. $7\sqrt{a}-6\sqrt{ab}+5\sqrt{b}-5\sqrt{ab}-\sqrt{a}$

58. $3.6a^m-5b^n+6.2b^n-2.5a^m$

Addition of Polynomials

The addition of polynomials is simply a matter of placing like terms in a column and combining them.

EXAMPLE. Add the following polynomials: $5a - 3b - 12c$ and $7b + 3a + 10c$.

SOLUTION.

$$\begin{array}{rl} & 5a - 3b - 12c \\ & \underline{3a + 7b + 10c} \\ \text{Sum,} & 8a + 4b - 2c \end{array}$$

To add two polynomials, place one polynomial under the other so that like terms are in vertical columns. Then add the columns.

Exercises

Find the sum of the following polynomials:

1. $4a - 3b + 5c,\ 8a - 9c + 5b$
2. $4r + 3s + t,\ 6r + 9s + 5t$
3. $6p - 4t + x,\ 8t - 7p + 6x,\ 7x - p - 4t$
4. $5b + 4c + 8d,\ 8b - 7c + 6d,\ 3b - 4c$
5. $7x - 2y + 5z,\ 8y - 9z,\ 7x - 5y$
6. $5x + 7y - 10,\ 8y - 2x + 3,\ 2x - 10y + 4$
7. $4a^2 - 5ab - 6b^2,\ 10ab - 6a^2 - 8b^2,\ 10b^2 - 3a^2 - 7ab$
8. $a - b + c,\ b - c - a,\ c - a + b,\ a - 2b - c$
9. $4ab - 5bc + 6ac,\ 6bc - 7ab - 8ac,\ 10ab - bc - ac$
10. $x^2 + y^2 - z^2,\ 3z^2 - 2x^2 - 4y^2,\ 4y^2 + x^2 + z^2$
11. $x^2 - 6x + 7,\ 8x - 15 - x^2,\ 4x^2 + 6x - 9,\ 7x + 10$
12. $a^2 - 2ab + b^2,\ 2ab + a^2 + b^2,\ 4a^2 - 4ab - b^2,\ a^2 - b^2$
13. $a + 2b,\ 3b - 4c,\ 5a - 7c,\ 3b + 2d$

Combine like terms:

14. $2(a + 5) + 3(a + 5) - 4(a + 5) + 6(a + 5)$
15. $3a + 4 + 2(3a + 4) - 3(3a + 4)$
16. $8\sqrt{a + b} - 5\sqrt{a + b} - 2\sqrt{a + b}$

Magic Squares — Just for Fun

Since the beginning of history, man has been interested in puzzles and games, many of them mathematical. One type of puzzle is the construction of magic squares.

At the right is a magic square of three rows. The numbers in each column, each row, and each diagonal add to 15.

8	1	6
3	5	7
4	9	2

You can make three-row magic squares for yourself by placing any integer you wish in place of 1 and then putting the next consecutive number in place of 2, and so on. Try it.

You can prove the statement above by using n for *any number*. What is the sum of the numbers in each row, each column, and each diagonal? Must n be an integer?

$n + 7$	n	$n + 5$
$n + 2$	$n + 4$	$n + 6$
$n + 3$	$n + 8$	$n + 1$

To make a magic square in which the sum of the numbers in each row, column, and diagonal is any number you please, put $3n + 12$ equal to that number and solve for n. (Where did we get $3n + 12$?)

This magic square is constructed on the same plan as the preceding one. Can you discover what the scheme is? What is the sum of the numbers in each row, column, and diagonal?

A magic square with any odd number of squares on each side can be constructed by using this same scheme.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Meaning of Subtraction

Do you know how clerks in stores make change? If you should buy an article for 63 cents and give the clerk a one-dollar bill, she probably would give you 2 pennies, a dime, and a quarter, and say "65 cents, 75 cents, one dollar." She finds the *difference* between 63 cents and a dollar by finding what must be *added* to 63 cents to make a dollar.

You know how to subtract 2 from 6. Would you know how to subtract -2 from 6? Before you can do this, you must think more carefully about the meaning of subtraction.

"Subtract 2 from 6" means: Find a number which added to 2 will make 6. The answer is 4.

In general, to subtract a from b means: To find a number which added to a will give b.

Hence, "Subtract -2 from 6" means: Find a number which added to -2 will give 6. The answer is $+8$, because we have to count ahead 8 from -2 to get 6.

"Subtract 6 from -2" means: Find a number which added to 6 will make -2. The answer is -8.

In the following examples the lower number is to be subtracted from the upper. What does that mean in each case?

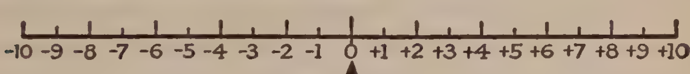
(*a*) $\begin{array}{r}7\\ \underline{3}\end{array}$ (*b*) $\begin{array}{r}7\\ \underline{-3}\end{array}$ (*c*) $\begin{array}{r}-7\\ \underline{-3}\end{array}$ (*d*) $\begin{array}{r}-7\\ \underline{3}\end{array}$

(*e*) $\begin{array}{r}3\\ \underline{7}\end{array}$ (*f*) $\begin{array}{r}3\\ \underline{-7}\end{array}$ (*g*) $\begin{array}{r}-3\\ \underline{-7}\end{array}$ (*h*) $\begin{array}{r}-3\\ \underline{7}\end{array}$

Give the answers to the examples above.

Subtracting by Use of the Number Scale

Using the number scale will give you further help in understanding what subtraction means.



Number scale

Subtract: (*a*) $\begin{array}{r}9\\ \underline{5}\end{array}$ (*b*) $\begin{array}{r}9\\ \underline{-5}\end{array}$ (*c*) $\begin{array}{r}-9\\ \underline{5}\end{array}$ (*d*) $\begin{array}{r}-9\\ \underline{-5}\end{array}$

The first example means: What number must be added to 5 to make 9? Start at 5 and see how far and in which direction 9 is. The answer is $+4$ because you must go 4 units to the right.

What does the second example mean? Start at -5; how far and in which direction is 9? The answer is $+14$. Why?

Do the third and fourth examples in the same way.

As you know, the number subtracted is called the ***subtrahend;*** the number from which it is subtracted is called the ***minuend.***

Exercises

1. Using the number scale, subtract the lower number from the upper number:

(*a*)	$\begin{array}{r} 9 \\ \underline{3} \end{array}$	(*d*)	$\begin{array}{r} -9 \\ \underline{3} \end{array}$	(*g*)	$\begin{array}{r} -3 \\ \underline{-9} \end{array}$	(*j*)	$\begin{array}{r} 0 \\ \underline{3} \end{array}$
(*b*)	$\begin{array}{r} -9 \\ \underline{-3} \end{array}$	(*e*)	$\begin{array}{r} 3 \\ \underline{9} \end{array}$	(*h*)	$\begin{array}{r} 3 \\ \underline{-9} \end{array}$	(*k*)	$\begin{array}{r} -3 \\ \underline{0} \end{array}$
(*c*)	$\begin{array}{r} 9 \\ \underline{-3} \end{array}$	(*f*)	$\begin{array}{r} -3 \\ \underline{9} \end{array}$	(*i*)	$\begin{array}{r} 3 \\ \underline{0} \end{array}$	(*l*)	$\begin{array}{r} 0 \\ \underline{-3} \end{array}$

Rule for Subtraction of Signed Numbers

If you understand what it means to subtract positive and negative numbers by using the number scale, you can discover a rule for subtracting more quickly.

Copy the two sets of examples in (1) and (2) below on the same piece of paper so that you can compare the examples and the answers. Note that the first set consists of subtraction examples and the second set consists of addition examples.

(1) *Subtract* the following numbers, using the number scale:

8	− 8	− 8	8	5	− 5	5	− 5
5	− 5	5	− 5	8	− 8	− 8	8

(2) *Add* the following numbers, using the rules you have learned:

8	− 8	− 8	8	5	− 5	5	− 5
− 5	5	− 5	5	− 8	8	8	− 8

(3) How do the upper numbers in the first set compare with the corresponding upper numbers in the second set?

(4) How do the lower numbers in the first set compare with the corresponding lower numbers in the second set?

(5) How do the answers in the first set compare with the corresponding answers in the second set?

(6) In the first example of the first set you *subtracted* 5 from 8 and got 3; in the first example of the second set you *added* − 5 to 8 and got the same answer. Make a similar statement concerning each of the other pairs of examples.

(7) To subtract + 5, you can add − 5. To subtract − 5, you can add + 5. To subtract + 8, you can add − 8. To subtract − 8, you can add + 8. You can do this because both ways, as you have seen, give the same answer. *You can change every subtraction example to an addition example.* This makes it easy to perform subtraction because you already know the rules for addition of signed numbers (page 148).

To subtract a positive number, add the negative number having the same absolute value.

For example, to subtract + 2, add − 2.

To subtract a negative number, add the positive number having the same absolute value.

For example, to subtract − 2, add + 2.

The two rules above may be stated as one, thus:

To *subtract* one signed number from another, change the sign of the subtrahend and *add* to the minuend.

(8) Using the rule above, subtract the lower number from the upper number in each of the following examples. In example (*a*) you should think: Add − 4 to + 7. In example (*b*) think: Add + 4 to − 7. And so on.

(*a*)	(*c*)	(*e*)	(*g*)
+ 7	− 7	− 4	+ 4
+ 4	4	− 7	− 7

(*b*)	(*d*)	(*f*)	(*h*)
− 7	+ 7	− 4	+ 4
− 4	− 4	7	+ 7

If you are at all confused in doing these subtraction examples, you should go back and practice the addition examples on page 153. When you understand addition thoroughly, you should have little difficulty with subtraction. Remember to change every subtraction example (mentally) to the corresponding addition example.

Exercises

In Exs. 1–76 subtract the lower number from the upper number by changing each example mentally to an addition example. Check your results by adding the remainder and the subtrahend to see if you get the minuend.

1. $\begin{array}{r} +4 \\ \underline{-6} \end{array}$
2. $\begin{array}{r} -6 \\ \underline{+4} \end{array}$
3. $\begin{array}{r} -6 \\ \underline{-4} \end{array}$
4. $\begin{array}{r} -4 \\ \underline{6} \end{array}$
5. $\begin{array}{r} -6 \\ \underline{4} \end{array}$
6. $\begin{array}{r} +4 \\ \underline{+6} \end{array}$
7. $\begin{array}{r} -4 \\ \underline{-6} \end{array}$
8. $\begin{array}{r} 6 \\ \underline{-4} \end{array}$
9. $\begin{array}{r} +8 \\ \underline{-5} \end{array}$
10. $\begin{array}{r} -9 \\ \underline{-3} \end{array}$
11. $\begin{array}{r} +2 \\ \underline{-6} \end{array}$
12. $\begin{array}{r} 0 \\ \underline{6} \end{array}$
13. $\begin{array}{r} 5 \\ \underline{+5} \end{array}$
14. $\begin{array}{r} 12 \\ \underline{0} \end{array}$
15. $\begin{array}{r} +12 \\ \underline{+7} \end{array}$
16. $\begin{array}{r} +7 \\ \underline{-10} \end{array}$
17. $\begin{array}{r} -2 \\ \underline{+2} \end{array}$
18. $\begin{array}{r} 6 \\ \underline{8} \end{array}$
19. $\begin{array}{r} 9 \\ \underline{5} \end{array}$
20. $\begin{array}{r} -5 \\ \underline{-5} \end{array}$
21. $\begin{array}{r} -2 \\ \underline{5} \end{array}$
22. $\begin{array}{r} 12 \\ \underline{24} \end{array}$
23. $\begin{array}{r} -3 \\ \underline{6} \end{array}$
24. $\begin{array}{r} -7 \\ \underline{-7} \end{array}$
25. $\begin{array}{r} +7 \\ \underline{+12} \end{array}$
26. $\begin{array}{r} -4 \\ \underline{+8} \end{array}$
27. $\begin{array}{r} -8 \\ \underline{-4} \end{array}$
28. $\begin{array}{r} -6 \\ \underline{+6} \end{array}$
29. $\begin{array}{r} +8 \\ \underline{-8} \end{array}$
30. $\begin{array}{r} +16 \\ \underline{-14} \end{array}$
31. $\begin{array}{r} -24 \\ \underline{12} \end{array}$
32. $\begin{array}{r} +15 \\ \underline{+15} \end{array}$
33. $\begin{array}{r} -9 \\ \underline{3} \end{array}$
34. $\begin{array}{r} -9 \\ \underline{10} \end{array}$
35. $\begin{array}{r} -10 \\ \underline{+10} \end{array}$
36. $\begin{array}{r} -2 \\ \underline{9} \end{array}$
37. $\begin{array}{r} 0 \\ \underline{-7} \end{array}$
38. $\begin{array}{r} -7 \\ \underline{0} \end{array}$
39. $\begin{array}{r} 12 \\ \underline{-12} \end{array}$
40. $\begin{array}{r} 15 \\ \underline{-3} \end{array}$
41. $\begin{array}{r} -28 \\ \underline{-10} \end{array}$
42. $\begin{array}{r} -54 \\ \underline{-14} \end{array}$
43. $\begin{array}{r} -64 \\ \underline{74} \end{array}$
44. $\begin{array}{r} 17 \\ \underline{27} \end{array}$

45. $\begin{array}{r}12\\2\\\hline\end{array}$	**47.** $\begin{array}{r}-13\\-3\\\hline\end{array}$	**49.** $\begin{array}{r}-2\frac{1}{2}\\1\frac{1}{4}\\\hline\end{array}$	**51.** $\begin{array}{r}-2\frac{1}{4}\\-3\frac{1}{2}\\\hline\end{array}$
46. $\begin{array}{r}13\\-3\\\hline\end{array}$	**48.** $\begin{array}{r}13\\3\\\hline\end{array}$	**50.** $\begin{array}{r}-2\frac{1}{4}\\+3\frac{1}{2}\\\hline\end{array}$	**52.** $\begin{array}{r}2\frac{1}{4}\\3\frac{1}{2}\\\hline\end{array}$

53. How do you subtract 5 from a number? (Answer: Add -5.)

54. How do you subtract -5 from a number?

55. $(+12)-(+7)=$ _?_

56. $(+12)-(-7)=$ _?_

57. $(-12)-(+7)=$ _?_

58. $(-12)-(-7)=$ _?_

59. $(+7)-(+12)=$ _?_

60. $(-7)-(+12)=$ _?_

61. $(-7)-(-12)=$ _?_

62. $(+7)-(-12)=$ _?_

63. $(-7)-(-7)=$ _?_

Subtraction of Monomials

From what you know about the subtraction of signed numbers you should be able to do the following exercises without difficulty. Change the sign of the subtrahend and proceed as in addition of monomials.

1. $\begin{array}{r}-12x\\-16x\\\hline\end{array}$	**7.** $\begin{array}{r}+y\\+y\\\hline\end{array}$	**13.** $\begin{array}{r}0\\-8x\\\hline\end{array}$	**19.** $\begin{array}{r}+\frac{7}{10}a\\+\frac{5}{10}a\\\hline\end{array}$
2. $\begin{array}{r}-6x\\10x\\\hline\end{array}$	**8.** $\begin{array}{r}+y\\-y\\\hline\end{array}$	**14.** $\begin{array}{r}-8x\\0\\\hline\end{array}$	**20.** $\begin{array}{r}-\frac{5}{8}m\\+\frac{1}{4}m\\\hline\end{array}$
3. $\begin{array}{r}-5.2k\\3.8k\\\hline\end{array}$	**9.** $\begin{array}{r}-y\\-y\\\hline\end{array}$	**15.** $\begin{array}{r}8x\\0\\\hline\end{array}$	**21.** $\begin{array}{r}0\\-8x\\\hline\end{array}$
4. $\begin{array}{r}+4x\\-x\\\hline\end{array}$	**10.** $\begin{array}{r}-y\\+y\\\hline\end{array}$	**16.** $\begin{array}{r}0\\8x\\\hline\end{array}$	**22.** $\begin{array}{r}-8y\\0\\\hline\end{array}$
5. $\begin{array}{r}-3r\\-12r\\\hline\end{array}$	**11.** $\begin{array}{r}+.82a\\-.18a\\\hline\end{array}$	**17.** $\begin{array}{r}12ab\\-12ab\\\hline\end{array}$	**23.** $\begin{array}{r}-\frac{3}{8}x^2y\\+\frac{3}{4}x^2y\\\hline\end{array}$
6. $\begin{array}{r}4s\\-4s\\\hline\end{array}$	**12.** $\begin{array}{r}-.82a\\-.18a\\\hline\end{array}$	**18.** $\begin{array}{r}15x^2\\-3x^2\\\hline\end{array}$	**24.** $\begin{array}{r}-9\frac{1}{2}x\\+\frac{2}{3}x\\\hline\end{array}$

Subtraction of Polynomials

To subtract one polynomial from another, place the one to be subtracted under the other so that like terms are in the same vertical column. Then subtract the columns.

EXAMPLE 1. Subtract $2a + b - c$ from $a - b + c$.

SOLUTION.

$$\begin{array}{rrrr} & a - & b + & c \\ & 2a + & b - & c \\ \hline \text{Difference,} & -a - & 2b + & 2c \end{array}$$

EXAMPLE 2. From $2a - 3b$ subtract $4b - 2c$.

SOLUTION.

$$\begin{array}{rrrr} & 2a - & 3b & \\ & & 4b - & 2c \\ \hline \text{Difference,} & 2a - & 7b + & 2c \end{array}$$

Exercises

Subtract the lower polynomial from the upper in each exercise:

1. $\begin{array}{r} 3a + 4 \\ \underline{2a + 9} \end{array}$

2. $\begin{array}{r} 7b - 5 \\ \underline{10b - 8} \end{array}$

3. $\begin{array}{r} r + 8s \\ \underline{r - 5s} \end{array}$

4. $\begin{array}{r} 7x - 3 \\ \underline{-2x - 9} \end{array}$

5. $\begin{array}{r} 3x^2 + 8y^2 \\ \underline{x^2 - 8y^2} \end{array}$

6. $\begin{array}{r} 5m^2 - 9n \\ \underline{8m^2 - 10n} \end{array}$

7. $\begin{array}{r} 8a - 9xy \\ \underline{-a + 9xy} \end{array}$

8. $\begin{array}{r} a - b + c \\ \underline{a + b - c} \end{array}$

9. $\begin{array}{r} x^2 + 2x + 1 \\ \underline{x^2 - 2x + 1} \end{array}$

10. $\begin{array}{ll} 2a & \\ \hline a & - b \end{array}$

11. $\begin{array}{llll} x^3 & & & + 8 \\ \hline x^3 & + x^2 & - 2x & \end{array}$

Add or subtract as indicated:

EXAMPLE 1. $2x + (3x - 4)$

SOLUTION. Add:

$$\begin{array}{rr} & 2x \\ & 3x - 4 \\ \hline \text{Sum,} & 5x - 4 \end{array}$$

EXAMPLE 2. $2x - (3x - 4)$

SOLUTION. Subtract:

$$\begin{array}{rr} & 2x \\ & 3x - 4 \\ \hline \text{Difference,} & -x + 4 \end{array}$$

12. $3x + (6x - 4)$

13. $3x - (6x - 4)$

14. $3x - (6x + 4)$

15. $3x - (-6x + 4)$

16. $(3a + 2b) + (5a - 6b)$

17. $(3a + 2b) - (5a - 6b)$

18. $(3a - 2b) - (5a + 6b)$

19. $(3a - 2b) + (5a + 6b)$

20. $(3a - 2b) - (-5a + 6b)$

21. $(3a - 2b) - (-5a - 6b)$

22. $(3a - 2b) + (-5a + 6b)$

23. $(6x - 5) - (-2x + 3)$

24. $(6x - 5) + (-2x + 3)$

---

25. From $3a - 2b + 5c$ take $-2a + 5b - c$.

26. Subtract $6x^2 - 3xy + y^2$ from $8x^2 + 5xy - y^2$.

27. From $6ab - 2ac + 5bc$ take $10ab - 2bc + 3ac$.

28. Subtract $x^2 - 2xy + y^2$ from $x^2 + 2xy + y^2$.

29. From $x^2 - 2xy + y^2$ take $x^2 + 2xy + y^2$.

30. From $a - b + c$ take $-a + b + c$.

31. From $a^2 + 6a + 9$ take $25 - 10a + a^2$.

32. From the sum of $x + 3y$ and $-3x - y$ subtract $x - y$.

33. Subtract $x - y - z$ from the sum of $2x^2 + 3y^2 - z^2$ and $4x^2 - 3y^2 + 5z$.

34. Take $a - b + 1$ from the sum of $a + c + 1$ and $a + b + 1$.

35. Simplify: $4x^2 - 8xy + 4y^2 - (9x^2 - 18xy + 9y^2)$.

36. By how much does $a + 2b - c$ exceed $3c - a + b$?

37. What must be added to $a^2 + 4$ to make $a^2 - 2a + 3$?

38. How much larger is $b + 2a - 3$ than $-a + 7$?

39. What must be subtracted from zero so that the remainder will be $a - b + c$?

40. Show that $(3x - 2y) - (x - 5y)$ is the same as $3x - 2y - x + 5y$.

Horizontal Addition and Subtraction: Single Terms in Parentheses

In algebra much of the addition and subtraction is done with the numbers written horizontally across the page instead of vertically in a column. There are certain rules to keep in mind, and some practice is needed, in working with numbers written horizontally.

The example $(+7) + (-8) + (+3)$ has the same meaning as $+7 - 8 + 3$. Note that the parentheses and the signs before the parentheses have been removed. The signs before the second and third parentheses indicate addition and in addition (unlike in subtraction) no changes of sign are made when the parentheses are removed. Hence, when given an example like the one above, in which all the signs before the parentheses are plus, rewrite it without the parentheses and without the plus signs before the parentheses.

The example $(+7) - (-8) - (+3)$ has the same value as the expression $+7 + 8 - 3$. Note that the parentheses and the signs before the parentheses have been removed. Note also that the signs of the numbers within the parentheses preceded by a minus sign have been changed. This is because the given example says to *subtract* -8 and to *subtract* $+3$. To do this, we change signs and *add* $+8$ and -3. In subtraction we change the sign of the subtrahend and add.

When a parenthesis is preceded by a minus sign, you may remove the parenthesis and the minus sign before it, provided you change the sign of the number within the parenthesis.

(1) Check the correctness of the following statements:

(*a*) $(+8) + (-3) + (+4) = 8 - 3 + 4 = 9$

(*b*) $(-7) + (+2) + (-3) = -7 + 2 - 3 = -8$

(*c*) $(+5) - (-3) - (+4) = +5 + 3 - 4 = 4$

(*d*) $-(-8) - (+2) + (-5) = +8 - 2 - 5 = 1$

(2) Find the sum of the numbers below.

$$(-8) - (-2) + (-3) - (+5) + (+9)$$

Exercises

Perform the indicated operations:

1. $(+8) + (+2)$
2. $(-8) - (+2)$
3. $(+8) - (-2)$
4. $(-8) + (-2)$
5. $(-8) - (-2)$
6. $(+8) + (-2)$
7. $(-8) + (+2)$
8. $(+8) - (+2)$
9. $(-2) - (+8)$
10. $(+2) - (-8)$
11. $(-2) - (-8)$
12. $(+2) - (+8)$
13. $8 + 2$
14. $8 - 2$
15. $-8 - 2$
16. $-8 + 2$
17. $2 - 8$
18. $-2 + 8$
19. $8a + 2a$
20. $8a - 2a$
21. $-8a - 2a$
22. $-8a + 2a$
23. $2a - 8a$
24. $-2a + 8a$
25. $(+7) + (+3) + (+5)$
26. $(+7) - (+3) - (+5)$
27. $(+7) - (+3) + (-5)$
28. $(-7) + (-3) - (+5)$
29. $(-7) - (-3) - (-5)$
30. $(+7) - (-3) + (-5)$
31. $(-6) + (+2) - (-3)$
32. $(-6) - (+2) + (-8)$
33. $(+6) - (-2) - (+8)$
34. $(+6) + (-2) - (+8)$
35. $+7 + 3 + 5 + 1$
36. $+7 - 3 + 5 - 1$
37. $-7 + 3 - 5 - 1$
38. $-7 - 3 + 5 - 1$
39. $-7 - 3 - 5 + 1$
40. $-7 - 3 - 5 - 1$
41. $-2 - 6 + 4 + 7$
42. $+2 + 6 - 4 - 7$
43. $+2 - 6 + 4 - 7$
44. $-2 - 6 - 4 - 7$
45. $-2 + 6 + 4 - 7$
46. $-2 + 6 - 4 - 7$
47. $4m - 2m + 2m + m$
48. $3x + 4x - x - 2x$
49. $4a - a - 2a + 4a$
50. $7p + 5p - 3p + 2p$
51. $2a - 3a - 7a - 5a$
52. $-4x + 3x - x + 6x$

53. $2n + 1 + 5n - 3$

54. $2n + 1 - 5n - 3$

55. $2n - 1 - 5n + 3$

56. $-2n + 1 + 5n + 3$

57. $2a + 3b + 4a + 2b$

58. $2a - 3b + 4a - 2b$

59. $(+8) + (+3) + (+7) + (+5)$

60. $(+8) - (+3) - (+7) + (+5)$

61. $(+8) - (-3) - (-7) + (+5)$

62. $(+8) - (+3) - (-7) + (+5)$

63. $(-8) - (+3) + (-7) - (+5)$

64. $(-8) + (-3) - (+7) - (-5)$

65. $(+8) + (+3) - (-7) - (+5)$

66. $(+8) - (+3) - (-7) - (-5)$

67. $(+8) - (-3) + (-7) - (+5)$

68. $(-8) + (-3) - (+7) + (-5)$

69. $(+4) - (+2) - (-6) + (+1)$

70. $(+4) - (-2) + (-6) - (-1)$

71. $(-4) - (+2) - (-6) + (+1)$

72. $(-4) + (-2) + (-6) - (-1)$

73. $(+8) + (-2) - (-6) + (-1)$

---

74. What is the value of $a - b$ when $a = 3$ and $b = -2$? Write: $(+3) - (-2) = \underline{\ ?\ }$.

75. What is the value of $a - b$ when $a = -3$ and $b = -2$?

76. What is the value of $a + b$ when $a = -3$ and $b = -2$? when $a = 3$ and $b = -2$?

77. What is the value of $a + b + c$ when $a = 2$, $b = 3$, and $c = -4$? when $a = -2$, $b = 3$, and $c = -4$?

78. What is the value of $a - b + c$ when $a = 3$, $b = 2$, and $c = -4$? when $a = 3$, $b = -2$, and $c = -4$?

Parentheses and Combining Terms

You already know that a parenthesis is used to indicate that a quantity of more than one term is to be treated as a single number. For example, to show that an expression like $2a - 3$ is to be subtracted from $5a$, we write: $5a - (2a - 3)$. You will now learn how to combine the terms (simplify the expressions) in addition and subtraction examples that involve parentheses. This is only an extension of what you learned in the preceding pages.

(1) Simplify: $7 - (6 - 3)$. Whenever it is possible to perform the operations indicated within a parenthesis, you should do so and then remove the parenthesis. Since $6 - 3 = 3$, you can rewrite $7 - (6 - 3)$ as $7 - 3$ and get the result 4.

(2) Simplify: $5a - (+2a - 3)$. In this problem you cannot perform the operations indicated within the parenthesis unless you know the value of a. The exercise means that you should subtract both $+2a$ and -3 from $5a$. How do you subtract $+2a$? (Add $-2a$.) How do you subtract -3? (Add $+3$.) You see that $5a - 2a + 3$ means the same thing as $5a - (+2a - 3)$.

SOLUTION.
$$\begin{aligned} &5a - (+2a - 3) \\ &= 5a - 2a + 3 \\ &= 3a + 3 \end{aligned}$$

Note that here the parenthesis is preceded by a minus sign and that in removing the parenthesis you also removed the sign before the parenthesis and changed the signs of all the terms within the parenthesis. This is in accord with the rule for subtraction, of changing the signs of the subtrahend and then adding.

The letter a in this example means *any number;* hence, the result should check for any value of a you choose. Suppose we let $a = 2$.

Then the given example, $5a - (+2a - 3) = 10 - (4 - 3) = 10 - 1 = 9$. The answer, $3a + 3 = 6 + 3 = 9$.

Since 9 is the value of the given example and also of the answer, the work checks for $a = 2$.

Now let us check by assigning a value of 5 to a.

The given example, $5a - (+2a - 3) = 25 - (10 - 3) = 18$. The answer, $3a + 3 = 15 + 3 = 18$. Here, again, the value of both the given example and the answer is the same, so that the work checks for $a = 5$.

Until you learn how to multiply signed numbers, you should use positive values of the letters in checking.

(3) Is there any difference in value between $5a + (2a - 3)$ and $5a + 2a - 3$? Check your answer, using any positive value you please for a.

(4) Rewrite $5a + (2a - 3)$ without the parenthesis and simplify the result.

SOLUTION.
$$\begin{aligned} &5a + (2a - 3) \\ &= 5a + 2a - 3 \\ &= 7a - 3 \end{aligned}$$

Note that here the parenthesis was preceded by a plus sign (indicating addition) and that you removed it and the plus sign without changing any signs. In addition we do not change signs.

Parentheses may be removed from algebraic expressions without changing the values of the expressions by following the rules given below.

When a parenthesis is preceded by a minus sign, you may remove it and the minus sign provided you change the signs of all the terms within the parenthesis.

When a parenthesis is preceded by a plus sign, you may remove it and the plus sign without changing the sign of any term within the parenthesis.

Exercises

Rewrite the following without parentheses, simplify the results, and check your answers, using any positive value of the letters:

1. $9 - (7a - 3)$
2. $9 - (7a + 3)$
3. $9 - (-7a - 3)$
4. $9 + (7a - 3)$
5. $7a + (2a - 3)$
6. $7a - (+2a - 3)$

7. $7a - (2a - 3)$
8. $7a - (2a + 3)$
9. $7a - (-2a + 3)$
10. $7a + (-2a + 3)$
11. $x - (3 + 2x)$
12. $x - (-3 + 2x)$
13. $x - (-3 - 2x)$
14. $(a + b) - b$
15. $-(a - b) + b$
16. $-(a - b) - b$
17. $5 + (2x - 3)$
18. $2a + (a - 7) + 9$
19. $b - (3b - 1) - 7$
20. $c - 8 - (7c - 5)$
21. $(a + b) - (a - b)$
22. $(a - b) + (a - b)$
23. $-(a - b) + (a - b)$
24. $-(a - b) - (a + b)$
25. $3a - (b + c)$
26. $3a - (b - c)$
27. $3a + (b + c)$
28. $3a + (b - c)$

---

29. $a^2 - (3 - a - a^2) - a$
30. $8 - (a - 3) + (4 + a)$
31. $+(x^2 + x + 1)$
32. $-(x^2 - 2x + 1)$

Other Symbols of Grouping

When it is necessary to group terms within a larger group of terms as in the expression $a - [b - (c + d)]$, either brackets [ ] or braces { } or both may be used. To simplify expressions of this kind, remove the innermost sign of grouping first.

EXAMPLE. Simplify: $3a - [8 - (2a - 7) + 5a]$

Removing the parenthesis: $= 3a - [8 - 2a + 7 + 5a]$

Removing the brackets: $= 3a - 8 + 2a - 7 - 5a$

Combining terms: $= -15$

Exercises

Simplify:

1. $+[r - (s - t)]$
2. $-[r - (s - t)]$
3. $a + \{b - (b - a)\}$
4. $2a - [4 + (2a - 8)]$
5. $3a + [5 - (6a + 9)]$
6. $7a - [2a - (b - 3a)]$
7. $4x - \{2y - (-y + 2x)\}$
8. $5x + [3x - (-2x - 3) + 2]$
9. $(3x + 4) - [2x - (7x - 5)]$
10. $-[6x + (-3x + 2)]$
11. $[5x - (-2x + 3) - 7]$
12. $-[2a - (3b - a) + (2a - 3b)]$

Placing Terms within a Parenthesis

As you continue in the study of algebra, you will find it necessary at times to place terms within a parenthesis. The rules for doing this without changing the value of the expression are the same as those for removing parentheses.

Terms of an expression may be placed within a parenthesis preceded by a minus sign, provided the signs of all the terms so placed are changed.

Terms of an expression may be placed within a parenthesis preceded by a plus sign without changing any signs.

EXAMPLE 1. Place the last two terms of $a - b + c$ within a parenthesis preceded by a minus sign.

$$a - b + c = a - (b - c)$$

Note that the sign of b which was originally minus becomes plus within the parenthesis, and the sign of c originally plus becomes minus.

EXAMPLE 2. Place the last two terms of $a - b + c$ within a parenthesis preceded by a plus sign.

$$a - b + c = a + (-b + c)$$

Note that the signs of b and c are not changed.

EXAMPLE 3. Place the last two terms of $a + b - c$ in a parenthesis preceded by a plus sign.

$$a + b - c = a + (b - c)$$

The correctness of each result may be checked by removing the parenthesis in your answer according to the rules on page 168. You should then have the given expression.

(1) Place the last two terms of the following expressions within a parenthesis preceded by a minus sign:

(a) $2a - 3b + c$ (b) $2a + 3b - c$

(2) Place the last two terms of $2a + 3b - c$ within a parenthesis preceded by a plus sign.

Exercises

1. Place the last two terms of each expression within a parenthesis preceded by a minus sign:

(*a*) $x - y + z$ (*c*) $x + y - z$ (*e*) $x^2 - 2x + 1$
(*b*) $x - y - z$ (*d*) $x - y - z$ (*f*) $x^2 + 2x - 1$

2. Place the last two terms of the expressions in Ex. 1 within a parenthesis preceded by a plus sign.

Equations Involving Parentheses[1]

You should have no difficulty in solving the equations on this page if you understand the work on page 167.

Solve and check: $7x - (x + 2) = 1$

SOLUTION.	$7x - (x + 2) = 1$
Removing parenthesis,	$7x - x - 2 = 1$
Combining terms,	$6x - 2 = 1$
Adding 2,	$6x = 3$
Dividing by 6,	$x = \frac{1}{2}$
CHECK.	$7x - (x + 2) = 7(\frac{1}{2}) - (\frac{1}{2} + 2) = 3\frac{1}{2} - 2\frac{1}{2} = 1$

Note that when we check equations we always perform the operations within the parentheses before removing them.

Exercises

Solve and check the following equations:

1. $n + (n - 3) = 1$
2. $n - (24 - n) = 8$
3. $4a - (2a + 1) = 11$
4. $7a + (3a - 5) = 45$
5. $5x + (2x + 3) = 73$
6. $8x - (7 - x) = 74$
7. $(2b + 3) - 5 = 6$
8. $(3b - 4) - 8 = 17$
9. $3n - (n + 7) + 7 = 10$
10. $n + (n + 7) = 8$
11. $3n - (n + 1) = 4$
12. $5a - (2a + 1) = 0$
13. $2a + (3a + 5) = 8$
14. $5x - (2x - 3) = 14$
15. $7x - (3x + 5) = 45$
16. $(4x + 5) - 32 = 3$

[1] TO THE TEACHER. See Note 15 on page 461.

Chapter Summary

In this chapter you have learned that adding signed numbers means finding the total effect of all the numbers given, whether they be positive or negative. You learned also that subtraction is really a form of addition, since subtracting a from b really means finding a number which when added to a gives b. From these meanings and by the use of the number scale you were able to discover rules for adding and subtracting positive and negative numbers. These rules were then applied in adding and subtracting algebraic numbers, both monomials and polynomials, and in discovering the rules that must be followed when parentheses are inserted or removed.

You should now review the rules for adding and subtracting signed numbers and for the removal and insertion of parentheses, making sure that you have these rules clearly and permanently in mind. You should understand the exact meaning of each of the following mathematical terms:

Number scale	Monomial	Coefficient
Absolute value	Binomial	Like terms
Polynomial	Trinomial	

Chapter Review

1. Start at 18 and keep adding − 3 until you reach − 18. Thus, 18, 15, 12, etc.

2. Start at 17 and keep adding − 3 until you reach − 19.

Number scale

3. Explain how you can add + 7 and − 3, using the number scale; also − 5 and + 2, − 3 and − 4.

4. What is the absolute value of + 6? of − 7?

5. Give the rule for adding two positive numbers; two negative numbers; a positive and a negative number.

6. How do you proceed in adding several numbers, some positive and some negative?

7. Add + 3 to each of the following numbers: 7, – 3, 12, 0, – 4, – 25, 16, – 35, 7.8, – .37.

8. Add the following:

(a)	(b)	(c)	(d)	(e)	(f)
$+3n$	$-a$	$-12x^2y$	$-13abc$	$+3x^3$	$-13xy$
$+8n$	$-a$	$+8x^2y$	$+13abc$	$-5x^3$	$-8xy$

9. Add – 5 to each of the following numbers: 6, – 2, 11, 0, – 3, – 21, 14, – 32, 6.2, – .45.

10. Add *a* and *b*.

11. "Subtracting 2 from 6" means "finding what number I must __?__ to __?__ to make __?__."

12. "Subtracting 6 from 2" means "finding what number I must __?__ to __?__ to make __?__."

13. How far is it on the scale from – 2 to + 3? In what direction? What is – 2 from 3? How much is (+ 3) – (– 2)?

14. Show on the number scale that the value of – 5 – (+ 4) is the same as – 5 – 4.

15. Give the rule for subtracting signed numbers.

16. Subtract – 3 from 7 by two methods, (1) on the number scale and (2) by means of the rule.

17. Subtract the lower number from the upper number:

(a)	(b)	(c)	(d)	(e)	(f)	(g)
10	-10	10	-10	6	$8x$	$-5y$
-4	-4	4	4	-2	$-3x$	$-8y$

(h)	(i)	(j)	(k)	(l)	(m)	(n)
$-15xy$	$12ab$	$7x^2$	$-7x^2$	$13p$	$14ab^2$	x
$-5xy$	$-4ab$	$-7x^2$	$7x^2$	$-p$	$-15ab^2$	$-3x$

18. Take – 10 *ab* from + 5 *ab*.

19. From – 4 *x* take – 6 *x*.

20. Take 17 xyz from 15 xyz.

21. Take $-4d$ from $+7d$.

22. Subtract a from b.

23. From 100 d take 10 c.

24. By how much does -8 exceed -2? 5 exceed -3? -2 exceed $+2$?

25. The sum of two numbers is 20. One of them is -5. What is the other?

26. Add the following:

(a)	(b)	(c)	(d)
-5	$+8$	$+5$	-10
$+7$	-15	$+7$	4
-9	-10	-9	-8
-3	$+16$	-3	3

27. Give the rules for removing a parenthesis from an expression without changing the value of the expression.

28. Perform the indicated operations:

(a) $(+6)-(-3)-(+5)+(-7)$
(b) $(-8)-(+5)+(+7)-(-9)$

29. Add:

$$\begin{array}{r} 3x^2+2xy-5y^2 \\ -7x^2-8xy \\ 4x^2-5xy+9y^2 \\ \hline \end{array}$$

30. Subtract:

$$\begin{array}{rrrr} 15x & -3y & +4z & \\ -12x & & +8z & -15t \\ \hline \end{array}$$

31. Add: $3a+2b+5c$, $-7a+8c$, $8b-13c$

32. Rewrite the following statements without parentheses and combine like terms:

(a) $b^2-(5-b-2b^2)+7$
(b) $8a+(-6a+3b)-(9a-4b)$

33. Give an example of each of the following: monomial, binomial, trinomial, polynomial.

34. Can an expression be both a binomial and a polynomial?

35. Can an expression be both a monomial and a polynomial?

Chapter VII

MULTIPLICATION AND DIVISION OF SIGNED NUMBERS

In multiplication and division of signed numbers, the rules for signs are of first importance, as in addition and subtraction. The rules in multiplication follow from consideration of addition and subtraction. The rules in division come from those in multiplication, since division is the inverse of multiplication.

The rules for signs are simple enough by themselves, but you need to exercise care in their use in actual computation work. Errors in signs in algebra do not arise generally from difficulties with the separate rules, but in their application to situations where all rules must be kept clearly in mind. Applications in this chapter should therefore receive particular attention.

How to Multiply Signed Numbers

In multiplying signed numbers, you will meet four combinations of signs as shown here:

(*a*) $(+4)(+3)$ (*c*) $(-4)(+3)$

(*b*) $(+4)(-3)$ (*d*) $(-4)(-3)$

(1) The first combination offers no difficulty. To multiply $+3$ by $+4$, you may add $+3$ four times. The result is $+12$. Hence $(+4)(+3) = +12$.

Add
$+\ 3$
$+\ 3$
$+\ 3$
$+\ 3$
$+\ 12$
$+3$ four times is $+12$

(2) The second sign combination is also easily understood. To multiply -3 by $+4$, you may add -3 four times. The result is -12. Hence $(+4)(-3) = -12$.

(3) The third combination, $(-4)(+3)$, is more difficult. What does it mean to multiply by -4? How can you take $+3$ minus four times? You can pick up a book four times. Could you pick it up minus four times?

Add
-3
-3
-3
-3
-12
-3 four times is -12

Many statements have no meaning, unless we shall agree on meanings for them, and this is one of them. To speak of multiplying a number by -4 (adding it minus four times) is to use words that have no meaning unless we give them a meaning. Mathematicians have agreed that to *multiply a number by* -4 shall mean to *subtract it four times.*

Using this definition of multiplication by -4, you can now multiply $+3$ by -4. To multiply $+3$ by -4, *subtract* $+3$ four times. Thus $-(+3)-(+3)-(+3)-(+3)=-3-3-3-3=-12$. That is, $(-4)(+3)=-12$.

Another way to look at this case is to remember that ab must equal ba (commutative law of multiplication, page 6). Then $(-4)(+3)$ must equal $(+3)(-4)$ and from the reasoning for the second combination of signs, you can see that this is -12.

(4) Now consider the fourth combination of signs, $(-4)(-3)$. This means, according to the reasoning developed in Ex. (3) above, that you should *subtract* -3 four times. Thus, $(-4)(-3)=-(-3)-(-3)-(-3)-(-3)=+3+3+3+3=+12$. Hence, $(-4)(-3)=+12$.

(5) To summarize,

$$(+4)(+3)=+12$$
$$(+4)(-3)=-12$$
$$(-4)(+3)=-12$$
$$(-4)(-3)=+12$$

(6) We have used the integers 3 and 4 in our illustrations. The law of signs in multiplication will be the same for any other integers.

The laws of multiplication for fractions and decimals that have positive or negative signs before them are the same as for integers. The same laws will hold for signed algebraic numbers.

(7) Complete the following statements concerning multiplication of signed numbers:

(*a*) A positive number multiplied by a positive number gives as a product a _?_ number.
(*b*) A positive number multiplied by a negative number gives as a product a _?_ number.
(*c*) A negative number multiplied by a positive number gives as a product a _?_ number.
(*d*) A negative number multiplied by a negative number gives as a product a _?_ number.

The product of two numbers which have like signs is positive.

The product of two numbers which have unlike signs is negative.

(8) Find the product: $(+3)(-2)(+4)$. We first multiply $(+3)$ by (-2), which equals (-6). Then $(-6)(+4) = -24$.

(9) Find the following products:

$(+3)(-2)(-4)$ $(-3)(+2)(-4)(-1)$
$(-3)(-2)(-4)$ $(+3)(-2)(-4)(+1)$
$(-3)(-2)(-4)(-1)$ $(+3)(-2)(-4)(-1)$

A product containing an even number of negative factors is positive and a product containing an odd number of negative factors is negative.

Exercises

Find the products of the following numbers:

1. $+6$ and -3
2. -5 and $+2$
3. -2 and -5
4. -5 and -2
5. $+2$ and -5
6. $+2$ and $+5$
7. $+3$ and -5
8. -3 and $+5$
9. -3 and -5
10. 4 and 0
11. -4 and 0
12. -4 and $+3$
13. 4 and -3
14. $-\frac{3}{4}$ and $+\frac{2}{3}$

15. Multiply each of the numbers in the next line by -2: $+5$, -6, $+4$, -7, -8, $+8$, -10, -4, -2, $+1$, -3.

16. Now multiply each of the numbers in Ex. 15 by $+4$; by -5; by $+6$; by -8; by -20; by 5.

Find the products as indicated:

17. $(+a)(+b)$
18. $(+a)(-b)$
19. $(-a)(+b)$
20. $(-a)(-b)$
21. $(+a)(+a)$
22. $(-a)(-a)$
23. $(a)(a)$
24. $(2)(-a)$
25. $(-a)(a)$
26. $(-2)(-a)$
27. $2(a+b)$
28. $2(a-b)$
29. $2(-a-b)$
30. $-2(a+b)$
31. $-2(a-b)$
32. 6×3
33. -6×3
34. -6×-3
35. 6×-3
36. $(-4)(-5)$
37. $(4)(-5)$
38. $-4(+5)$
39. $4(+5)$
40. $(+3)(+5)$
41. $(+6)(-2)$
42. $(+10)(-2\frac{1}{2})$
43. $(+6)(-9)$
44. $(-2)(-5)$
45. $(+8)(-\frac{1}{4})$
46. $(+12)(+6)$
47. $12(-6)$
48. $-12(6)$
49. $(-10)(5)$
50. $8(-2.5)$
51. $12(-9)$
52. $(+8)(-9)$
53. $(+9)(-8)$
54. $(-7)(-9)$
55. $(+7)(-8)$
56. $(-6)(-9)$
57. $(-7)(-6)$
58. $\frac{2}{3}(-12)$
59. $\frac{4}{5}(+20)$
60. $-2(-5)$
61. $(-8)(-0.8)$
62. $(+6)(-4)$
63. $(-4)(-7p)$
64. $4(3p)$
65. $-4(6p)$
66. $(2a)(-3a)$
67. $(2a)(-3b)$
68. $(-2a)(-5a)$
69. $(-2a)(-5b)$
70. $(+5)(+4)(-2)$
71. $(-5)(+4)(-2)$
72. $(+5)(-4)(-2)$
73. $(-5)(-4)(-2)$
74. $(-5)(-4)(+2)$
75. $(+1)(+1)(+1)(+1)$

76. $(-1)(-1)(-1)(+1)$

77. $(-1)(-1)(-1)(-1)$

78. $4(2a-3)$

79. $-4(2a-3)$

80. $-4(2a+3)$

81. $a(a-b)$

82. $-a(a+b)$

83. $-a(a-b)$

84. $3(n^2-2n+1)$

85. $-3(n^2-2n+1)$

Multiply as indicated:

86. $(892)(78)$

87. $(904)(703)$

88. $(854)(-600)$

89. $(-7649)(-580)$

90. $(-48)(-10)$

91. $(-62)(100)$

92. $(963)(-1000)$

93. $(3.8)(10)$

94. $(4.92)(-10)$

95. $(-6.374)(100)$

96. $(\frac{1}{2})(-\frac{1}{2})$

97. $(-\frac{2}{3})(-\frac{3}{2})$

98. $(\frac{5}{6})(\frac{3}{10})$

99. $(36)(-\frac{7}{8})$

100. $(-\frac{9}{16})(-36)$

101. $(9\frac{1}{4})(432)$

102. $(-\frac{5}{6})(\frac{14}{35})(-\frac{12}{7})$

103. $(-7\frac{1}{2})(-8\frac{3}{4})(-3\frac{1}{5})$

104. $(3.4)(-70.75)$

105. $(.067)(472.5)$

106. $(-8.93)(-103.7)$

107. $(.79)(-654.91)$

Applying Knowledge of Signed Numbers[1]

The examples below call for the addition, subtraction, and multiplication of signed numbers and indicate how to carry out these operations. Work carefully through the problems and compare your answers with those given in the solution.

EXAMPLE 1. Find the value of $2a - 3b$ when $a = 2$ and $b = -3$.

SOLUTION. $2a = 2(2) = 4$ and $3b = 3(-3) = -9$. Hence $2a - 3b = 4 - (-9) = 4 + 9 = 13$.

EXAMPLE 2. Find the value of $2a - 3b - 4c$ when $a = 3$, $b = -5$, and $c = 2$.

SOLUTION. $2a = 2(3) = 6$, $3b = 3(-5) = -15$, and $4c = 4(2) = 8$. Hence, $2a - 3b - 4c = 6 - (-15) - 8 = 6 + 15 - 8 = 13$.

[1] TO THE TEACHER. See Note 16 on page 461.

EXAMPLE 3. Find the value of $a^2 - b^2$ when $a = -3$ and $b = -4$.

SOLUTION. $a^2 = (-3)(-3) = 9$, $b^2 = (-4)(-4) = 16$. Hence $a^2 - b^2 = 9 - 16 = -7$.

EXAMPLE 4. Find the value of $a - 3(b - c)$ when $a = 2$, $b = 5$, and $c = -7$.

SOLUTION. Find the value of the numbers in the parenthesis before multiplying. $b - c = 5 - (-7) = 12$. Hence, $a - 3(b - c) = 2 - 3(12) = 2 - 36 = -34$. (Remember that the multiplication of -3 by 12 must take place before the 2 is added.)

EXAMPLE 5. If $y = 3 - 2x$, what is the value of y when $x = -5$?

SOLUTION. $2x = -10$. Hence, $y = 3 - (-10) = 13$.

EXAMPLE 6. Find the value of $2a^2b^3c$ when $a = -2$, $b = -3$, and $c = -4$.

SOLUTION. $a^2 = 4$, $b^3 = -27$, and $c = -4$. Hence $2a^2b^3c = 2(4)(-27)(-4) = 864$.

EXAMPLE 7. Simplify $3a - 4(2a - 1)$.

SOLUTION. To multiply $2a - 1$ by -4, think: $-4(2a) = -8a$ and $(-4)(-1) = +4$. Then $3a - 4(2a - 1) = 3a - 8a + 4 = -5a + 4$.

In Ex. 7, the a, of course, means *any number*. It should check, therefore, for any number you choose. Choose $a = 3$ and see if the original expression and your answer have the same value.

CHECK. $3a - 4(2a - 1) = 9 - 4(6 - 1) = 9 - 4(5) = 9 - 20 = -11$. $-5a + 4 = -15 + 4 = -11$.

(8) Find the value of $2a - 3b$ when $a = 5$ and $b = 4$.

(9) Find the value of $2a - 3b$ when $a = 5$ and $b = -4$.

(10) Find the value of $2a - 3b$ when $a = -5$ and $b = -4$.

(11) What is the value of $a^2 - b^2$ when $a = 4$ and $b = -3$?

(12) What is the value of $a - 3(b - c)$ when $a = 2$, $b = -5$, and $c = -3$?

(13) If $y = 3 - 2x$, what is the value of y when $x = -3$?

(14) Find the value of $2a^2b^3c$ when $a = 2$, $b = -5$, and $c = 4$.

(15) Simplify $5a - 3(4a - 7)$.

Exercises

Find the value of the following expressions when $a = 3, b = -2, c = -5$, *and* $d = 4$:

1. $2a + 3b$
2. $2a - 3b$
3. $-2a - 3b$
4. $-2a + 3b$
5. $4b + 2c$
6. $4b - 2c$
7. $-4b + 2c$
8. $-4b - 2c$
9. $3c - 2d$
10. $-3c - 2d$
11. $-3c + 2d$
12. $3c + 2d$
13. ab
14. $-ab$
15. bc
16. $-bc$
17. cd
18. $-cd$
19. ad
20. $-ad$
21. $2ab$
22. $2bc$
23. $2cd$
24. $2ac$
25. $-2ac$
26. $-2cd$
27. $-2bc$
28. $-2ab$
29. a^2
30. b^2
31. $a^2 + b^2$
32. $a^2 - b^2$
33. $b^2 + c^2$
34. $b^2 - c^2$
35. $(a + b)^2$
36. $(a - b)^2$
37. $(b + c)^3$
38. $(b - c)^3$
39. $(a + b)(c - d)$
40. $(a - b)(c + d)$
41. $2(a + c)$
42. $3(b + c)$
43. $2(a - c)$
44. $3(b - c)$
45. $d + 2(a - c)$
46. $b - 3(c - d)$

Find the value of y in each of the following equations when the values of x are as given:

47. $y = 2x - 5$; for $x = 3$ and for $x = -4$
48. $y = 6 - 3x$; for $x = 5$ and for $x = -4$
49. $y = 2 + 5x$; for $x = 4$ and for $x = -3$
50. $y = 3 - 2x$; for $x - 7$ and for $x = -3$

Simplify the following expressions:

51. $3a + 4(2a + 1)$
52. $3a + 4(2a - 1)$
53. $3a - 4(2a - 1)$
54. $3a - 4(2a + 1)$
55. $2b - 5(4 - 3b)$
56. $3b + 5(3 - 2b)$
57. $-7b - 3(-4 + 7b)$
58. $-2b - 4(-1 - b)$

How to Divide Signed Numbers

By using the fact that division is the inverse of multiplication, you can discover for yourself the laws of signs in division.

(1) You know that $\frac{15}{5}$ is 3 because $3 \times 5 = 15$. How do you know that $\frac{24}{6} = 4$? How do you know that $\frac{b^2}{b} = b$? that $\frac{4a}{2} = 2a$?

(Division by 0 is impossible. Try to find the answer to $\frac{3}{0}$. There is no number which multiplied by 0 will give 3.)

(2) Show why the following quotients are correct:

(*a*) $\frac{18}{6} = 3$ (*b*) $\frac{18}{-6} = -3$ (*c*) $\frac{-18}{6} = -3$ (*d*) $\frac{-18}{-6} = 3$

(3) Complete the following statements or principles about the division of positive and negative numbers. Make use of the statements in Ex. (2) above.

When you divide a positive number by a positive number, you get a _?_ number.

When you divide a positive number by a negative number, you get a _?_ number.

When you divide a negative number by a positive number, you get a _?_ number.

When you divide a negative number by a negative number, you get a _?_ number.

(4) If the signs of dividend and divisor are alike, what is the sign of the quotient?

(5) If the signs of dividend and divisor are unlike, what is the sign of the quotient?

(6) Using the rules which you made in Exs. (4) and (5) above, state the answers to the following:

(*a*) $8 \div 4$ (*b*) $8 \div -4$ (*c*) $-8 \div -4$ (*d*) $-8 \div 4$

The quotient of two numbers which have like signs is positive.

The quotient of two numbers which have unlike signs is negative.

Exercises

Perform the indicated divisions and check by multiplication:

1. $\frac{+6}{+2}$	**7.** $\frac{20}{5}$	**13.** $\frac{4a}{2}$	**19.** $\frac{-8a}{a}$
2. $\frac{-6}{-2}$	**8.** $\frac{20}{-5}$	**14.** $\frac{4a}{-2}$	**20.** $\frac{-8a}{-a}$
3. $\frac{+6}{-2}$	**9.** $\frac{0}{5}$	**15.** $\frac{-4a}{-2}$	**21.** $\frac{b^2}{b}$
4. $\frac{-6}{+2}$	**10.** $\frac{0}{-5}$	**16.** $\frac{-4a}{2}$	**22.** $\frac{b^2}{-b}$
5. $\frac{-20}{5}$	**11.** $\frac{6}{0}$	**17.** $\frac{8a}{a}$	**23.** $\frac{-b^2}{b}$
6. $\frac{-20}{-5}$	**12.** $\frac{-6}{0}$	**18.** $\frac{8a}{-a}$	**24.** $\frac{-b^2}{-b}$

25. Divide each of the following numbers by -5:

$+10,\ -5,\ 0,\ -20,\ +20,\ -12,\ -5x,\ +5y$

26. Below are three lines of numbers. Cover the second and third with a strip of paper. Divide each of the numbers of the first line by -2. Now see if you obtained the numbers written in the second line.

$-6,\ +18,\ +12,\ -12,\ 0,\ -24,\ -9,\ -6x,\ +6x$

$+3,\ -9,\ -6,\ +6,\ 0,\ +12,\ +4.5,\ +3x,\ -3x$

$-1,\ +3,\ +2,\ -2,\ 0,\ -4,\ -1.5,\ -x,\ +x$

Cover the lower line and write the quotients obtained by dividing the numbers in the middle line by -3. See if you have the numbers appearing in the lower line.

Divide as indicated:

27. $\frac{3}{4} \div -\frac{3}{4}$	**32.** $-6\frac{2}{3} \div 3\frac{3}{4}$
28. $21 \div -\frac{3}{4}$	**33.** $-3241 \div -10$
29. $\frac{7}{10} \div 14$	**34.** $785 \div -100$
30. $-\frac{3}{7} \div -\frac{8}{21}$	**35.** $3.2 \div 1000$
31. $-13\frac{1}{2} \div 3\frac{5}{6}$	**36.** $-4590 \div 45$

37. $-5963 \div -67$ **38.** $8526 \div 98$

39. $-896.4 \div .69$ correct to the nearest tenth.

40. $-5.981 \div -5.33$ correct to the nearest hundredth.

41. $7.016 \div .493$ correct to the nearest hundredth.

---

42. $(+3)(-5) + (-7)(+2)$

43. $(-3)(-4) - (-7)(-3)$

44. $(7)^2 + 2(7)^2(-3)$

45. $(4)^3 - 5(3)(-2)$

46. $(\sqrt{36})(3)^3 + (7)(-4)$

47. $(-5)^3(-2) - (3)(\sqrt{100})$

More Practice with Signed Numbers

Find the value of the following expressions for each of the given values of x. When an expression contains a parenthesis, evaluate the terms in the parenthesis first.

1. $3x;\ x = -2,\ x = -\frac{2}{3},\ x = -\frac{3}{2}$

2. $-5x;\ x = 4,\ x = 5,\ x = -\frac{1}{2}$

3. $\frac{2x}{3};\ x = -5,\ x = +6$

4. $-\frac{3x}{2};\ x = 6,\ x = -4,\ x = -7$

5. $\frac{3x}{5};\ x = -2,\ x = +7$

6. $2x + 3;\ x = -3,\ x = 5$

7. $3x - 5;\ x = -2,\ x = \frac{1}{2}$

8. $-3x + 2;\ x = 5,\ x = -3$

9. $-2x + 3;\ x = \frac{1}{3},\ x = \frac{3}{5}$

10. $5 - 2x;\ x = 5,\ x = -3$

11. $4 - 3x;\ x = -\frac{2}{3},\ x = \frac{3}{5}$

12. $2(x - 3);\ x = 4,\ x = -2$

13. $3(5 - 2x)$; $x = 3$, $x = -2$

14. $5 + 3(x - 5)$; $x = 7$, $x = -7$

15. $7 - 2(x + 1)$; $x = -3$, $x = 4$

16. $7x + 2(x - 3)$; $x = 4$, $x = -1$

17. $5x - 3(x + 2)$; $x = -3$, $x = 2$

18. $x + 2(3x - 4)$; $x = 5$, $x = -5$

19. $x - 2(3x - 4)$; $x = 5$, $x = -5$

Solve and check the following equations:

EXAMPLE. $8x - 5(x + 2) = 11$

SOLUTION. $8x - 5x - 10 = 11$

Combining terms, $3x - 10 = 11$

Adding 10, $3x = 21$

Dividing by 3, $x = 7$

CHECK. $8x - 5(x + 2) = 56 - 5(9)$
$= 56 - 45$
$= 11$

20. $2(x - 5) + 3 = 13$

21. $7(a - 2) - 2 = -2$

22. $x - (7 - 2x) = 2$

23. $7c + (c + 5) = 7$

24. $5y - 3(1 + y) = 5$

25. $-3(7 - 5a) = 9$

Find the value of each of the following expressions for each of the given values of x:

26. $\frac{2x}{3}$; $x = -\frac{4}{5}$, $x = -\frac{3}{2}$

27. $-\frac{3x}{2}$; $x = \frac{3}{2}$, $x = -\frac{4}{5}$

28. $\frac{3x}{5}$; $x = \frac{1}{2}$, $x = -\frac{3}{4}$

29. $-3x + 2$; $x = -\frac{1}{2}$, $x = \frac{3}{5}$

30. $2(x - 3)$; $x = \frac{2}{3}$, $x = -\frac{3}{2}$

31. $5 + 3(x - 5)$; $x = \frac{3}{4}$, $x = -\frac{3}{4}$

32. $7 - 2(x + 1)$; $x = \frac{2}{3}$, $x = -\frac{3}{2}$

33. $x - 2(3x - 4)$; $x = \frac{3}{2}$, $x = -\frac{3}{2}$

Solve and check the following equations:

34. $4(a - 1) - 2(3 + a) = 0$

35. $4(x + 3) - 2(x + 4) = 6$

36. $x - 3(7 - x) = 5$

37. $9b - (9 + 3b) - 2b = -1$

38. $4x + 3(x - 4) - (x - 12) = 0$

Expressing Statements Algebraically

1. State the following as equations:
   (*a*) Three times a number is 15.
   (*b*) Three more than a number is 15.
   (*c*) Three less than a number is 15.
   (*d*) Three less than four times a number is 15.
   (*e*) Four times a number is the same as six more than the number.
   (*f*) Four more than four times a number is the same as twice the sum of the number and four.

2. What is the value in cents of n dimes? of b quarters? of c half dollars? of $3n$ nickels? of $(2n + 3)$ dollars?

3. What is the number of yards in $(n + 1)$ feet?

4. Using A for the amount, p for principal, i for the interest, r for the rate in hundredths, and t for the time, write as a formula the fact that the amount equals the principal plus the product of the principal, rate, and time.

5. If $6a^2$ represents the area of a rectangle, what represents the area of a rectangle twice as large?

6. If $2a + b$ represents the perimeter of a triangle, what represents three times this perimeter?

7. If Mary is n years old and her mother is 3 years less than twice as old, how old is the mother in terms of n?

8. Sound travels in air at the rate of 1100 ft. per second. How far will it travel in t seconds? How many seconds will it take to travel d ft.?

Official photograph, U. S. Army Air Corps

A ground crew at work. The 40-hour inspection.

9. If for every hour in the air an airplane requires 16 man-hours of mechanical overhaul and care, how many hours will be required for overhaul and care of a planes of the same kind, each of which is in the air an average of b hours?

10. If an automobile travels at the rate of 35 miles an hour, what distance will it cover in 10 hours? in n hours? in $3n$ hours? in $(n + 2)$ hours?

11. Express the distance covered by an automobile in 10 hours if its rate is —

(*a*) $5n$ miles an hour

(*b*) $(n + 3)$ miles an hour

(*c*) $(2n - 5)$ miles an hour

(*d*) $3(n + 5)$ miles an hour

12. A slow train travels at the rate of r miles an hour and another train travels 20 miles an hour faster. Express —

(*a*) the rate of the fast train, (*b*) the distance traveled by each in 5 hours, and (*c*) the fact that the sum of the distances covered by the trains is 300 miles.

13. The length of a rectangle is 5 feet more than its width, w. State algebraically that its perimeter is 95 feet.

14. If a plane uses 18 gal. of gasoline in a flight of m miles, how much will it use in a flight of n miles?

15. A pursuit plane has an initial climb of 2400 feet per minute. How far does it climb in t seconds?

Problems in Words

1. The sum of three numbers is 72. The first is three times the second and the third is twice the result of subtracting 6 from the second. What are the numbers?

2. A man spent $58 for a watch, a chain, and a cardcase. He paid $2 more for the chain than for the case and twice as much for the watch as for the chain. How much did he pay for each?

3. The length of a basketball court is 20 ft. more than its width, and the perimeter is 240 ft. Find the dimensions.

4. Here is an old puzzle made easy by algebra: A bottle and a cork cost $1.10. The bottle cost a dollar more than the cork. What did each cost?

5. A has $100 more than B, and C has as much as A and B together. C has $900. How much have A and B?

6. In triangle ABC, angle A is 35° more than angle B, and angle C has as many degrees as angle A and angle B together. How many degrees are there in each angle? (The sum of the three angles of a triangle is 180°.)

7. In exchange for 4 dollar bills Fred received equal numbers of each of the following kinds of coins: nickels, dimes, and quarters. How many of each kind did he receive?

8. A bomber costs four times as much as a fighter plane. If five fighters and two bombers cost $650,000, find the cost of one plane of each kind.

9. Two planes set out at the same time from two airports 825 miles apart and fly toward each other with speeds of 150 m.p.h. and 180 m.p.h. respectively. After how many hours will they meet?

10. The Consolidated B-24 bomber has a wingspread of twice the Lightning two-motored fighter and 6 ft. more. The total stretch of 3 bombers and 2 fighters placed wing to wing is 434 ft. Find the wingspread of one bomber and one fighter.

11. A plane flies for 3 hours at a uniform speed and for 4 hours at a speed 20 m.p.h. greater. Find the original speed if the distance traveled was 850 miles.

12. At 2 P.M. a train traveling at 50 miles an hour started from A toward B. At 3 P.M. a train traveling at 40 miles an hour started from B toward A. A is 410 miles from B. At what time will the two trains meet?

13. Two boys riding bicycles start from the same place. One rides at the rate of 8 miles an hour and the other at 5 miles an hour. They go in the same direction, but the faster boy starts one hour later than the other. In how many hours from the start of the slower boy will they be 14 miles apart? (Which boy will be ahead?)

Chapter Summary

In this chapter, you learned that you may expect any one of four different sign combinations in multiplying and dividing. You not only discovered what multiplication and division mean for each of these combinations of signs, but you also learned how to multiply and divide in each case.

You found that —

(*a*) The product or the quotient of two numbers having like signs is positive.

(*b*) The product or the quotient of two numbers having unlike signs is negative.

(*c*) In evaluating an expression containing parentheses, you perform the operations indicated within the parentheses first.

(*d*) In evaluating an expression calling for a series of operations, after the parentheses have been evaluated, you perform the multiplications and divisions before the additions and the subtractions.

You were able to put the knowledge gained in this chapter into practice in evaluating algebraic expressions where the letters represented both positive and negative numbers, where terms in parenthesis were included, and where operations were called for in various orders. You will find this information very useful in many phases of your work in algebra.

You should now turn back and do at least one typical example in each set of exercises.

Chapter Review

1. What is the law of signs in multiplying signed numbers?

2. How can you tell by inspection whether the product of several signed numbers is positive or negative?

3. The square of a positive number is a _?_ number.

4. The square of a negative number is a _?_ number.

5. The cube of a positive number is a _?_ number.

6. The cube of a negative number is a _?_ number.

Evaluate the following:

7. $(+1)(-1)$
8. $(-1)(-1)$
9. $(+2)(-8)$
10. $(-2)(-8)$
11. 7^2
12. $(-4)^2$
13. $(+6)^3$
14. $(-2)^3$
15. $(-1)^4$
16. $(-1)^5$
17. $(-2)(-3)(-4)$
18. $(-2)(+3)(-4)$

19. In evaluating an expression calling for a series of operations, after the parentheses have been evaluated, _?_ and _?_ should be performed before _?_ and _?_.

Evaluate:

20. $(+3)(-2)+(-7)(+3)$
21. $(-3)(-4)+(-7)(-3)$
22. $(-3)(-4)-(-7)(-3)$
23. $4^2 + 2(5)^2(-1)$

24. If you were to evaluate $3a^2b - b$ when $a = 5$ and $b = 2$, what would you do first, second, etc.?

25. If you were to evaluate $2(ab^2)^2 - 3(-a)^3$, when $a = 5$ and $b = 2$, what would you do first, second, etc.?

Find the value of each of the following when $a = 2$, $b = -3$, $c = -4$, *and* $d = 5$:

26. $3a + 2b$	**32.** bc	**38.** $-3bc$	**44.** ab^2
27. $3a - 2b$	**33.** $3ab$	**39.** a^2	**45.** $a^2 + b^2$
28. $-3a - 2b$	**34.** $3ad$	**40.** b^2	**46.** $a^2 - b^2$
29. $-3a + 2b$	**35.** $3bc$	**41.** a^3	**47.** $5(a - b)$
30. ab	**36.** $-3ab$	**42.** b^3	**48.** $5(b + c)$
31. ad	**37.** $-3ad$	**43.** a^2b	**49.** a^2b^2

Find the value of each of the following when $a = 3$, $b = -4$, $c = -5$, *and* $d = 4$:

50. $5a - 3ab + 2b$

51. $4c - 3d + 5a$

52. $3a^2 - ab$

53. $3c^2 - 2c - 5$

54. $3c^2d - 2cd^2 + 2$

55. $2b^3 - 3b^2 - 2b$

56. $6 - 2(b + d)$

57. $a + 2(3c - d)$

58. $d + 3(a - c)$

59. $(d + 3)(a - c)$

60. What is the law of signs for dividing signed numbers?

Find the value of each of the following:

61. $+8 \div +4$

62. $-12 \div -3$

63. $-12 \div +6$

64. $+18 \div -2$

65. $\frac{-20}{-5}$

66. $\frac{36}{12}$

67. $\frac{-24}{+6}$

68. $\frac{+10}{-5}$

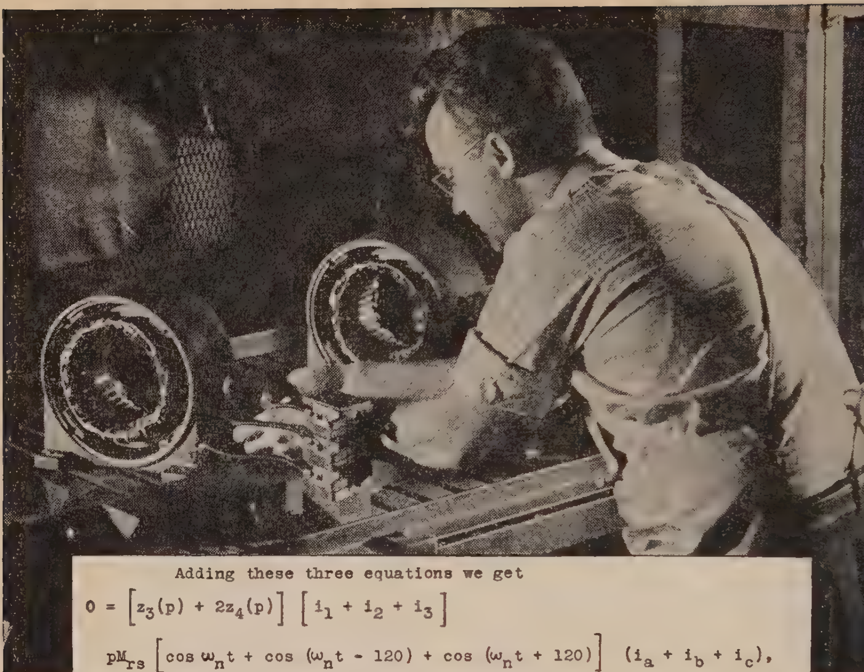
69. $\frac{13}{0}$

70. $\frac{0}{24}$

71. Indicate the sum of $5a$, $6b$, and $-c$.

72. Indicate that $3c - 4d$ is to be subtracted from a.

73. Indicate that the product of -5 and $a + b$ is to be added to $3x$.



Adding these three equations we get

$$0 = \left[z_3(p) + 2z_4(p)\right] \left[i_1 + i_2 + i_3\right]$$

$$pM_{rs} \left[\cos \omega_n t + \cos (\omega_n t - 120) + \cos (\omega_n t + 120)\right] \quad (i_a + i_b + i_c),$$

$$\cos \omega_n t + \cos (\omega_n t - 120) + \cos (\omega_n t + 120) = \qquad \text{———}$$

$$\cos \omega_n t \left[1 + \cos 120 + \cos 120\right] + \sin \omega_n t \left[\sin 120 - \sin 120\right] = 0,$$

therefore: $i_1 + i_2 + i_3 = 0,$

Similarly from the three stator equations

$$i_a + i_b + i_c = 0.$$

Now making use of these relations the rotor equations are solved for each of the rotor currents in terms of the stator currents.

General Electric Co.

The man is testing the windings of wire in the stator, or non-moving part, of a new electric motor. Much mathematical computation underlies the designing and building of a motor. The typewritten matter above is part of a page from an engineer's analysis of an electric motor. You are not familiar with much of that symbolism, but the work suggests how extensively algebraic calculations are used in modern industry and the importance of being able to carry them out correctly and efficiently.

Chapter VIII

WAYS OF MULTIPLYING AND DIVIDING LITERAL NUMBERS

Sometimes a child beginning the study of arithmetic in the lower grades does not understand why he should learn to add, subtract, multiply, and divide arithmetic numbers. At the moment he does not see a need to work with numbers in that way, and he fails to realize the importance of the fundamental operations. But if that boy or girl does not master the work, he has trouble all along in solving everyday problems that are easy for those who have taken pains to become efficient in computing.

Learning these fundamental operations in algebra is just as important for future work in algebra, and in other mathematics, as it is in the case of arithmetic. You need to understand each step and practice its use by doing the exercises, until you can operate easily and accurately with algebraic numbers.

Exponents in Multiplication

This section reviews some of the things you already know about exponents and leads you to a rule for multiplying numbers with exponents.

(1) Write the following in a simpler way:

(*a*) 2×2 (*b*) $m \times m \times m$ (*c*) $(r)(r)(r)(r)$

(2) When you see 2^3, the exponent 3 tells you how many times 2 is to be taken as a factor. What does 2^3 mean? What is its value? Do you remember that the 2 is called the *base?*

(3) The expressions 2^2, 2^3, 2^4, and 2^5 are powers of 2. They are respectively the second, third, fourth, and fifth powers of 2. What are their values? What is the first power of 2? The answer is 2. The exponent 1 is not written when you have a first power. 2^4, 2^3, and 2^2 are read respectively: 2 to the fourth power, 2 to the third power, and 2 to the second power. What is another way of reading 2^2 and 2^3?

(4) Read the following expressions: (*a*) $2\,a^2b^3c^4$, (*b*) $5\,a^7b^4$, (*c*) $9\,x^6y^3z$.

(5) Can you tell what the product of n^2 and n^3 is? Try it; then check your answer by reading further.

n^2 means $(n)(n)$; n^3 means $(n)(n)(n)$. Hence $n^2 \times n^3$ means $(n)(n) \times (n)(n)(n)$ or n^5. That is, $n^2 \times n^3 = n^5$.

(6) Use the same reasoning in finding the following products:

$$a^2 \times a^4,\ a \times a^2,\ a^3 \times a^4,\ a \times a^5,\ 2^2 \times 2^3,\ 10^3 \times 10^2.$$

(7) Do you see a short-cut rule for multiplying such numbers?

To find the exponent of the product of two (or more) powers having like bases, add the exponents of these bases.

Thus, $a^2 \times a^4 = a^6$ and $a \times a^2 = a^3$. In the language of algebra the rule is:

$$\boldsymbol{a^m \times a^n = a^{m+n}}$$

A common error made by pupils is that $2^2 \times 2^3$ is 4^5. If 2 is taken as a factor twice and then again as a factor three times, is it 4 or 2 that is taken as a factor five times? $2^2 \times 2^3$ means $(2)(2) \times (2)(2)(2)$, which is 2^5.

Exercises

Multiply as indicated:

1. $a^2 \times a^3$

2. $n^3 \times n^4$

3. $b \times b^2$

4. $(a)(a)$

5. $b^3 \times b$

6. $b^2 \times b^3$

7. $(-b)(-b)$

8. $b(-b)$

9. $(-a^5)(a^2)$

10. $(x)(-x^2)$
11. $(-y^2)(y^2)$
12. $(-y^2)(-y^2)$
13. $(a)(a^2)(a^3)$
14. $3 \times 2p$
15. $p^3 \times p^2$
16. $a \times b$
17. $(a^3)(a^2)(b)$
18. $a^4 \times a$
19. $(x^5)(x^2)(x)$
20. $(-a)(-a)(-a)$
21. $(-a)(-a)(-a)(-a)$
22. $2^2 \times 2^3$
23. 3×3^2
24. $(2)(2^2)(2^3)$
25. $(3)(3)(3^2)$
26. $(4^2)(4^3)$
27. $(5)(5^2)$
28. $(2^2)(3^2)$
29. $(2^2)(2^4)$
30. $a^m \times a^n$

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31. $a^{n-2} \cdot a$
32. $3^x \cdot 3^2$
33. $x^{2a} \cdot x^{3a}$
34. $(a^n)^2$
35. $3^n \cdot 3^n$
36. $x^a \cdot y^a$

Multiplying a Monomial by a Monomial

Multiplying a monomial by a monomial depends upon the principles $ab = ba$ and $abc = (ab)c$ or $a(bc)$ or $(ac)b$. That is, in multiplication the order in which the factors are taken makes no difference in the result. Neither does it make any difference which of the factors are associated.

Study the following example to make sure you understand these statements:

$$2 \times 3 \times 4 \times 5 \times 6 = 720.$$

Now change the order of the factors in any way you please — for example,

$2 \times 6 \times 5 \times 3 \times 4$ — and the product is still 720.

Now associate any two or three or four of the factors — for example,

$(2 \times 3) \times (4 \times 5) \times 6 = 6 \times 20 \times 6$ — and the result is still 720.

Also $2 \times (4 \times 3) \times (6 \times 5) = 2 \times 12 \times 30 = 720.$

Or $(2 \times 3 \times 4) \times (5 \times 6) = 24 \times 30 = 720.$

Try other combinations.

From the preceding principles we get the rule for multiplying monomials:

> **To multiply two monomials, change the order of the factors so that those factors which will combine may be associated. Then combine these factors by multiplication.**

EXAMPLES.

(a) $5 \times 2\,ab = (5 \times 2) \times ab = 10\,ab$

(b) $a \times 2\,ab = 2 \times (a \times a) \times b = 2\,a^2b$

(c) $3\,b \times 2\,ab = (3 \times 2) \times a \times (b \times b) = 6\,ab^2$

(d) $(2a^2b)(-4a^2b^3c) = (2 \times -4)(a^2 \times a^2)(b \times b^3)(c) = -8a^4b^4c$

(e) Find the product of $2\,a^2$ and $3\,ab^3$. Check your result.

SOLUTION. $2\,a^2 \times 3\,ab^3 = (2 \times 3)(a^2 \times a)(b^3) = 6\,a^3b^3$.

Since a and b are any numbers, this should check for any values of a and b you choose. Let $a = 2$ and $b = 3$.

CHECK. $2\,a^2 \times 3\,ab^3 = 2(4) \times 3(2)(27) = 8 \times 162 = 1296$

$$6\,a^3b^3 = 6(8)(27) = 1296$$

(Note that number 1 should be avoided in checking examples with exponents. If you let $a = 1$, then a^2, a^3, a^4, etc., all equal 1 and you cannot tell by this means whether there has been a mistake in exponents.)

In multiplying monomials, the intermediate step which we have shown is usually omitted; that is, the association of the proper factors is done mentally. Are the following examples correct?

$2 \times 3\,a = 6\,a$	$2\,a \times 3\,a^2 = 6\,a^3$
$a \times 3\,a = 3\,a^2$	$a^2 \times 3\,a^2 = 3\,a^4$
$-2 \times 3\,a^2 = -6\,a^2$	$a^3 \times -3\,ab = -3\,a^4b$

Exercises

Multiply as indicated:

1. $2 \times 3\,b$
2. $b \times 3\,b$
3. $a \times ab$
4. $b \times ab$

5. $(+2)(+5b)$
6. $(-2)(+5b)$
7. $(-a)(-5b)$
8. $(+b)(+5b)$
9. $(-b)(-5b)$
10. $3y^2 \times y$
11. $3y^2 \times y^2$
12. $3x \cdot 5x^2$
13. $-3x \cdot 5x^2$
14. $-3x \cdot -5x^2$
15. $(7n)(4n)$
16. $(4n^3)(-2n)$
17. $(2a)(-3b)$
18. $(3bc)(4b)$
19. $(-3x^2)(4)$
20. $(-3x^2)(4y)$
21. $(-4a^3x)(-5a^2)$
22. $(-4a^2)(-2ab)$
23. $(3a)(ab)$
24. $(a^2b)(ab^2)$
25. $(ab^2)(ab^2)$
26. $6a^2b \cdot a^3$
27. $-7x^2 \cdot -6x$
28. $-4x^2y \cdot 5y$
29. $(6a^2b)(a^3)$
30. $(5a^3b^2)(4a^2bc^2)$
31. $(4a^2)(-1)$
32. $(2a)(-3b)$
33. $(-5x)(-4y)$
34. $(ab^2x)(-a^2bc)$
35. $(3b)(2b^3)$
36. $(-7a)(2a)$
37. $(-5x)(4x)$
38. $(-3n)(-n)$
39. $(4x^3)(5x^2)$
40. $(2x^2)(-3x^3)$
41. $(10p)(-3p^3)$
42. $(-2y)(-y)(y)$
43. $(a)(-a)(a)(-a)$
44. $(-2)(-3x)(-4y)$
45. $(4x)(0)(2x)$
46. $(-x^2)(-5)(-5)$
47. $(2x^3)(-2x)(y^2)$
48. $(4n)(\frac{1}{4}mn^2)$

49. Distinguish carefully between $2(ab)$ and $2(a+b)$. What is the answer to each?

50. What is the value in cents of $2n$ quarters? of $(2+n)$ quarters?

51. What is the cost of $3n$ dozen eggs at $4p$ cents a dozen?

52. How far will an auto travel in $5n$ hours at a rate of $3n$ miles an hour?

Multiply as indicated:

53. $(-\frac{5}{8}x^2)(-4x^2y)$
54. $(-c)(-ab)(-abc)$
55. $(0.2r^3)(5r^2)$
56. $(1.4x)(5y^2)$
57. $(0.4ab^2)(-ab)(-ab)$
58. $(5a^2b)(-3ab^2)(a^3b)$
59. $(3ab^2)(-5a^2b^3)(-6ab^2)$
60. $(7a^xy^b)(3a^yy)$
61. $(a^{n-1}b)(a^2b^n)$
62. $4c^a \cdot -5c^a$
63. $(x^n)^2$
64. $(-3^n)^2$
65. $(-x)(-x)(-y)(-y)(-1)$

Raising a Monomial to a Power

Just as s^3 means $s \times s \times s$, $(2s)^3$ means $2s \times 2s \times 2s$ or $8s^3$. Note that the 2 as well as the s has been cubed or raised to the third power. When an exponent appears outside a parenthesis, it applies to every *factor* of the expression within the parenthesis. Thus, $(3x^2y)^2$ equals $(3x^2y)(3x^2y) = 9x^4y^2$.

Raise the following to the indicated powers:
$(a^2)^2$, $(a^2)^3$, $(x^3)^2$, $(b^4)^2$, $(b^4)^3$, $(b^4)^5$, $(c^5)^2$, $(c^5)^3$, $(c^5)^6$.
The answers are respectively a^4, a^6, x^6, b^8, b^{12}, b^{20}, c^{10}, c^{15}, c^{30}.
Do you see that an easy way to do examples like these, in which you have to raise a number with an exponent to a power, is to multiply the two exponents?

To raise an indicated product to a power, raise each of the factors to that power. To raise a number with an exponent to a power, multiply the exponent by the index of the power.

EXAMPLE. $(-2a^2b^3c^4)^3 = -8a^6b^9c^{12}$

Note that $(ab)^2$ is the same as a^2b^2 but $(a+b)^2$ is not the same as a^2+b^2. Check this statement with any values of a and b you please. ab is an indicated product, while $a+b$ is an indicated sum. *The above rule does not apply to indicated sums.*

Exercises

Raise the following to the powers indicated:

1. $(n^2)^2$ 2. $(n^2)^3$ 3. $(c^2)^2$ 4. $(c^2)^3$ 5. $(x^5)^2$ 6. $(x^5)^3$

7. $(n^3)^5$
8. $(-n^2)^2$
9. $(-n^2)^3$
10. $(2x)^2$
11. $(3xy)^2$
12. $(-2s)^3$
13. $(-2s)^4$
14. $(-3b^2)^2$
15. $(-3ab)^2$
16. $(-3ab^2)^3$
17. $(2a^2b)^2$
18. $(-2a^2b)$
19. $(2a^2b)^3$
20. $(-2a^2b)^3$
21. $(2x^2y^3)^3$
22. $(4a^2b^3c)^2$
23. $(4a^2b^3c)^3$
24. $(4a^2b^3c)^4$
25. $(-5a^2bc^4)^2$
26. $(-3a^2bc^4)^5$
27. $(x^2)^n$
28. $(x^8)^n$
29. $(a^n)^n$
30. $(a^2)^{n+1}$

31. What is the value of each of the following? Do each example in two ways: (1) by evaluating what is in the parenthesis and then raising your answer to the indicated power, and (2) by raising what is in the parenthesis to the power first and then evaluating your answer.

(*a*) $(3x)^2$ when x is 5? 2? $\frac{1}{3}$?
(*b*) $(3xy)^2$ when x is 1 and y is 3?
(*c*) $(2a^2b)^2$ when a is 2 and b is 4?
(*d*) $\frac{4}{3}\pi r^3$ when π is $\frac{22}{7}$ and r is 3?
(*e*) $(2x^3y^2)^2$ when x is 5 and y is 3?

Raise to the indicated powers and then multiply:

32. $(2x^3)(3x^2)^2$
33. $(-5y^2)(y^2)^3$
34. $(xy^2)(x^2y)^2$
35. $(a^2b)^3(ab)$
36. $(2xy)^2(5x^3y)^2$
37. $(2ab)^2(5ab)^2$
38. $(x^2y)^2(3x^2y^2)^3$
39. $(a^2b^2)(x^2y)^2$
40. $(-2)^2(0)(3)^2$
41. $(2x)^2(3x)(0)(-1)^2$
42. $(5xy)(2ab)^3$
43. $(3x)^2(2x)^2$

Evaluate the following:

44. $2(10)^3$
45. $3(10)^4$
46. $1.3(10)^2$
47. $10^2 \cdot 10^3$
48. $2 \cdot 10^2 \cdot 10$
49. $5(10)^3$
50. $1.2(10)^2$
51. $.75(10)^3$
52. $1.354(10)^4$

Multiplying a Polynomial by a Monomial

You learned at the beginning of this course that $a(b + c) = ab + ac$ and you have made use of this fact many times in simple cases. For example, $2(3 + 4) = 6 + 8$ or $3(a - b) = 3a - 3b$. State in words what $a(b + c) = ab + ac$ means. It means, "If you multiply one number by the sum of two other numbers you will get the same result as if you multiply the first by the second, then the first by the third, and then add these two products."

This same principle applies, no matter how many numbers you have or what numbers you have. Thus —

$$a(b + c + d + e + \cdots) = ab + ac + ad + ae + \cdots$$

(The dots mean "and so forth.")

In words, the rule may be stated:

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and add the results.

EXAMPLE. $2a^2b(a - b) = 2a^3b - 2a^2b^2$

Exercises

Multiply as indicated:

1. $3(a + b)$
2. $3(a - b)$
3. $3(-a - b)$
4. $-3(a + b)$
5. $-3(a - b)$
6. $-3(-a - b)$
7. $a(a - b)$
8. $-a(a - b)$
9. $b(a + b)$
10. $-b(a + b)$
11. $5(2x - 3)$
12. $-5(3 - 2x)$
13. $3(x^2 + 2y^2)$
14. $-3(x^2 - 2y^2)$
15. $x(x^2 + 2y^2)$
16. $y(x^2 - 2y^2)$
17. $4(2f - 3i)$
18. $5(6x - 4)$
19. $6(2x - 3)$
20. $-7(x - 3)$
21. $a(a - 3)$
22. $x(x - 4)$
23. $-x(2x - 3)$
24. $-p^2(p - 2)$

25. $a(a^2 - b)$
26. $-a(a^2 - b)$
27. $2x(3x - 5)$
28. $y^2(2y - 3)$
29. $3(3 - x^2)$
30. $x(x^2 - x)$
31. $-2y(10 - 5y)$
32. $-xy(x - y)$
33. $xy(x - y)$
34. $5x(1 - x^2)$
35. $2ab(a - b)$
36. $p^2(1 - p^2)$
37. $p(r - s)$
38. $2ab(a + b)$
39. $2ab(ab)$
40. $x(x + 3)$
41. $x(3x)$
42. $x^2(x - 3)$
43. $x^2(-3x)$
44. $x^2(x^2 - 2y^2)$
45. $x^2(-2x^2y^2)$
46. $5x^3(x^2 - y^2)$
47. $x(x^2 - x - 1)$
48. $p(p - 4 - s)$
49. $-3y(x - 2y + y^2)$
50. $2m(5m^2 - 6m + 2)$
51. $-3a(5a^2 - 6a + 7)$
52. $d(a + b - c)$

Multiply as indicated and then combine similar terms:

53. $2x + 3(x - 2)$
54. $2x - 3(x - 2)$
55. $2x - 3(-x + 2)$
56. $2x + 3(-x + 2)$
57. $4n - 3(2n + 5)$
58. $4n + 3(2n - 5)$
59. $4n + 3(-2n - 5)$
60. $5(3x - 7) - 2$
61. $-5(3x + 7) - 2$
62. $-5(7 - 3x) - 5x$
63. $2a + 3b - 5b(a - b)$
64. $2a - 3b + 5b(b - a)$
65. $2(2a + 3b) - 3(3a - 2b)$
66. $-2(2a - 3b) + 3(3a + 2b)$
67. $2a(a - b) + 3a(2a + b)$
68. $2a(a - b) - 3a(2a + b)$
69. $-2a(a - b) + 3a(2a - b)$
70. $2a(a - b) - 3a(-2a + b)$
71. $x^2 - 3(x^2 + 1)$
72. $x^2 + 3(x^2 - 1)$
73. $x^2 - 3(-x^2 + 1)$
74. $x^2 - x(-x^2 + 1)$

Find the values of the following expressions for each of the values of the letters given. (Evaluate the parentheses first.)

75. $2x + 3(x - 2)$; $x = 3, -2, \frac{1}{2}$
76. $2x - 3(x - 2)$; $x = -4, 3, 2\frac{1}{2}$
77. $2x - 3(-x + 2)$; $x = 5, -3\frac{1}{3}$
78. $5(3x - 7) - 2$; $x = 6, -1\frac{2}{3}$
79. $-5(3x + 7) - 2$; $x = -1, 0.6$
80. $-5(7 - 3x) - 5x$; $x = 8, -2$
81. $3x - 4(x - 2)$; $x = 7, -5$
82. $(3x - 4)(x - 2)$; $x = 7, -5$

Solve and check the following equations:

83. $3(2x - 9) - 5(10 - x) = 0$
84. $3(5x - 2) + 2(x - 5) - 5x = 8$
85. $4(3x + 7) - 2(x - 5) - 7x = -1$
86. $9n - 2(1 + 4n) = 7$
87. $4(5n - 6) - 2(n - 7) + 5(-3n + 2) = 0$
88. $-3x + 2(5x - 3) + 7x = +8$
89. $6(2 - y) - 3(y - 4) - 5(3 - 2y) = 1$
90. $4(2x - 3) - 5(x - 1) + 6(2x + 1) = 24$

Multiplying a Polynomial by a Polynomial

If you were asked to multiply $(6 - 4 + 2)$ by $(3 + 2)$, you would combine the terms within the parentheses and then multiply. But if your two polynomials were $x^2 - 2xy + y^2$ and $x + y$, you could not combine the terms. You would have to find some other way to multiply these two numbers. By analyzing the first example, we shall see how to do the second.

Suppose the numbers in $(6 - 4 + 2)$ and $(3 + 2)$ are general numbers that we cannot combine, and that we wish to multiply them. Instead of combining terms within the parentheses and then multiplying (giving $4 \times 5 = 20$), we multiply each term of $(6 - 4 + 2)$ first by 3 and then by 2, and add the results.

This gives $(18 - 12 + 6) + (12 - 8 + 4) = 12 + 8 = 20$, which we know is the correct answer.

This suggests the method of multiplying a polynomial by $x + y$. First multiply each term of the polynomial by x, then by y, and add the results. In algebraic symbols the method is shown as follows:

$$(x + y)(x^2 - 2xy + y^2) = x(x^2 - 2xy + y^2) + y(x^2 - 2xy + y^2)$$

You already know how to carry out the multiplication and addition on the right-hand side of this equation.

A convenient form for the multiplication and addition is shown at the right. Place the two polynomials as indicated; first multiply by x and then by y, placing like terms in a column. Then add the columns.

$$\begin{array}{rrrr} x^2 & -\,2xy & +\,y^2 & \\ x & +\,y & & \\ \hline x^3 & -\,2x^2y & +\,xy^2 & \\ & x^2y & -\,2xy^2 & +\,y^3 \\ \hline x^3 & -\,x^2y & -\,xy^2 & +\,y^3 \end{array}$$

In this example $x^2 - 2xy + y^2$ is the *multiplicand* and $x + y$ is the *multiplier*.

If the terms of the multiplicand and the multiplier are given to you in a hit-or-miss order, they should first be arranged according to *descending* (or *ascending*) powers of some letter. For example, $3x + 12x^2 + 16 + 3x^3$ should be arranged as $3x^3 + 12x^2 + 3x + 16$ (descending order of powers of x) or as $16 + 3x + 12x^2 + 3x^3$ (ascending order). The multiplicand and the multiplier should be in the same order, either both ascending or both descending.

(1) Multiply as indicated:

(*a*) $\begin{array}{l} 2a^2 - 3ab + 5b^2 \\ \underline{3a - 2b} \end{array}$ (*b*) $\begin{array}{l} 3x - 5 \\ \underline{2x + 3} \end{array}$ (*c*) $\begin{array}{l} a + b \\ \underline{c + d} \end{array}$

To check, use any values of the letters you wish. The value of the multiplicand multiplied by the value of the multiplier should equal the value of the product.

To multiply one polynomial by another polynomial, multiply each term of the first one by each term of the second one and add the results.

Exercises

Multiply as indicated:

1. $(x + 3)(x - 4)$
2. $(2a + 3b)(a - b)$
3. $(2a - 5)(3a + 2)$
4. $(a + b)(c + d)$
5. $(a - b)(c + d)$
6. $(2x + 3)(3x - 2)$
7. $(a + 3)(a + 2)$
8. $(x - 3y)(2x + 3y)$
9. $(a - b)(c - d)$
10. $(a + b)(c - d)$
11. $(3b - 4c)(3b + 4c)$
12. $(3b - 4c)(2c + 3d)$
13. $(4x - 5)(4x + 5)$
14. $(7n - 3)(7n + 3)$
15. $(3x - 4)(3x - 4)$
16. $(2x + 4y)(x - 3y)$
17. $(n + 7)(n - 5)$
18. $(2a - 3b)(a - 4b)$
19. $(5x - 3y)(3x + 4y)$
20. $(a + b)(a + b)$
21. $(a + b)(a - b)$
22. $(a^2 + b)(a^2 - 2b)$
23. $(a^2 + b^2)(a^2 - 2b^2)$
24. $(x^2 - y^2)(x^2 + y^2)$
25. $(m^2 - 2n^2)(m^2 - 2n^2)$
26. $(4s - 3)(2s^2 + 3)$
27. $(a^3 + b^3)(3a^3 - b^3)$
28. $(2xy - 1)(2xy + 1)$
29. $(x - 3)^2$
30. $(2x + 3y)^2$

31. What is the difference in meaning between the two expressions below? Find the value of each when $x = 4$.

(*a*) $(2x - 3)(3x + 4)$ (*b*) $2x - 3(3x + 4)$

Find the values of these expressions when n *is* $3, -3, 2\frac{1}{2}$, *and* $-2\frac{1}{2}$. *(Evaluate the parentheses first.)*

32. $3n - 2(n + 5)$
33. $(3n - 2)(n + 5)$
34. $(n + 4)(n - 3)$
35. $n + 4(n - 3)$
36. $(3n - 2)^2$
37. $(2n + 3)(2n - 3)$

Multiply as indicated:

38. $(x^2 - 2x + 1)(x - 1)$
39. $(x^2 + 2x + 1)(x + 1)$
40. $(x^2 - 2x + 1)(x + 1)$
41. $(x^2 + 2x + 1)(x - 1)$

42. $(x^2-6xy+9y^2)(x-3y)$

43. $(x^2-6x+8)(x+2)$

44. $(x^2-7xy+12y^2)(x-3y)$

45. $(n^3-5n^2+2)(n^2-2n)$

46. $(x^2-xy+y^2)(x+y)$

47. $(x^2+xy+y^2)(x-y)$

48. $(2-3a^2-a)(a-1)$

49. $(c-2c^2+3)(2+c)$

50. $(x^2+x+1)(x^2-x+1)$

51. $(x^3+3x^2+2x-1)(x-1)$

52. $(x^3-3x^2+3x-1)(x-1)$

53. $(5y^2-2-3y)(3-y)$

54. $(3a^2+2-5a)(4a-3)$

55. $(a^3-5+2a-3a^2)(2+a)$

Expressions Involving Multiplication of Binomials

It is easy to make mistakes in signs in working with the expressions below. Study the example so that you will know how to avoid these mistakes.

Multiply as indicated and combine like terms:

$$5x-(2x+3)(3x-4)$$

This expression says that you should *subtract* the product of the two quantities in parentheses from $5x$. To do this, first multiply $(2x+3)$ by $(3x-4)$. Then, when you have found the product, put it in a parenthesis and substitute it in the expression in place of the indicated product, $(2x+3)(3x-4)$, in order that the expression may say the same thing now as it did originally. Thus:

$$\begin{aligned} 5x-(2x+3)(3x-4) &= 5x-(6x^2+x-12) \\ &= 5x-6x^2-x+12 \\ &= -6x^2+4x+12 \end{aligned}$$

To check, substitute any number you wish, say 2, for x in the given example and in the answer.

CHECK. $5x-(2x+3)(3x-4)=10-(7)(2)=10-14=-4$
$-6x^2+4x+12=-24+8+12=-4$

In similar expressions where you have to add or subtract the product of binomials, perform the operations in the following order:

1. Carry out the indicated multiplications.
2. Substitute the product thus found in a parenthesis.
3. Perform the additional operations indicated.

Exercises

Multiply as indicated and combine like terms:

1. $5n - (2n - 3)(3n - 2)$
2. $5n + (2n - 3)(3n - 2)$
3. $5n + 2n - 3(3n - 2)$
4. $5n + 2n + 3(3n - 2)$
5. $13x + (2x - 5)(5x - 2)$
6. $13x - (2x - 5)(5x - 2)$
7. $13x - 2x + 5(5x - 2)$
8. $2a - (a + 3)(a - 3) + a^2$
9. $2a + (a - 3)(a + 3) + a^2$
10. $(2a + 3b)(5a - 7b) + 9ab$
11. $(2a - 3b)(5a + 7b) - 9ab$
12. $3a(2a - b) - 5a(a + b)$
13. $3a(2a - b) + 5a(a + b)$
14. $(x - 3)(x + 2) + (x + 4)(x - 7)$
15. $(x - 3)(x + 2) - (x + 4)(x - 7)$

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16. $5(3x - 4)(x + 7)$
17. $(2x - 3)(2x + 3) - (4x - 5)(4x + 5)$
18. $(2x - 1)^2 + (2x + 1)^2$
19. $(2x - 1)^2 - (2x + 1)^2$
20. $x - 3x + 2(x + 4)(x - 7)$
21. $3(5 - 2x)(4 - 5x) - 2(7 - 2x)$

22. $2b(3a - 6b) - 3a(a + 2b)$

23. $(3x - 4)^2 + (3x - 5)^2$

24. $(3x - 4)^2 - (3x - 5)^2$

25. $(2a + b)(2a - b) - (a + b)(a - b)$

26. $(7x - 2y)(6x - 5y) - 3x(x - 4y)$

27. $(4p - 3)(3p + 5)(2p - 7)$

28. $(R + r)(R - r) - (R - r)^2$

29. $4ab - (a + 2b)^2$

30. $(a + b)^3$

31. The width of a rectangle is 3 inches less than the side n of a square; the length is 5 inches more than the side of the square. The area of the square is 7 square inches more than the area of the rectangle. Express in terms of n —

(*a*) the length and the width of the rectangle.

(*b*) the area of the square.

(*c*) the area of the rectangle.

(*d*) Write the second sentence of the problem as an equation.

(*e*) Solve the equation to find the dimensions of the square and the rectangle.

Exponents in Division

You have learned to perform multiplication like $n^3 \times n^5$ by adding the exponents. Since division is the inverse of multiplication, you can easily see that if $n^3 \times n^5 = n^8$, then $n^8 \div n^5 = n^3$ or $n^8 \div n^3 = n^5$.

Another way to analyze this division example is shown below.

$$\frac{n^8}{n^5} = \frac{\cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdot n \cdot n \cdot n}{\cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdot \cancel{n} \cdot \cancel{n}} = n^3$$

(1) Using either of the above methods of reasoning, find the following quotients:

(*a*) $\frac{n^5}{n^3}$ (*b*) $\frac{n^4}{n^3}$ (*c*) $\frac{n^3}{n}$ (*d*) $\frac{2^5}{2^2}$

(2) Can you write a short-cut rule for dividing such numbers?

To find the exponent of the quotient of two powers having like bases, subtract the exponent in the divisor from the exponent in the dividend.

Thus, $\frac{n^5}{n^2} = n^3$ and $\frac{n^3}{n} = n^2$. Stated algebraically, the rule is:

$$\frac{a^m}{a^n} = a^{m-n}$$

(3) Is $2^{10} \div 2^5 = 2^2$? Is $2^6 \div 2^2 = 1^4$? Is $x^6 \div x^2 = x^3$? Explain your answers.

If you continue your study of algebra, you will learn about *logarithms*. By means of logarithms difficult multiplications and divisions are done by simply adding and subtracting exponents. In such work a table of powers of 10 is used. (See the table of powers of 2 at the top of the next page.)

Exercises

Divide as indicated:

1. $x^2 \div x$
2. $y^9 \div y^4$
3. $c^4 \div c$
4. $2^6 \div 2^3$
5. $3^5 \div 3$
6. $7^6 \div 7^2$
7. $6^3 \div 6$
8. $3^5 \div 3$
9. $d^7 \div d^6$
10. $b^5 \div (-b^2)$
11. $a^7 \div (-a^3)$
12. $-n^4 \div (-n)$
13. $x^{10} \div x^2$
14. $x^{10} \div x^5$
15. $x^8 \div x^3$
16. $\frac{c^6}{c^2}$
17. $\frac{n^5}{n}$
18. $\frac{n^5}{n^4}$
19. $\frac{a^5}{a^3}$
20. $\frac{b^2}{b}$
21. $\frac{b^3}{b}$
22. $\frac{-a^5}{a^4}$
23. $\frac{b^3}{-b}$
24. $\frac{6b}{3}$
25. $\frac{2^5}{2^2}$
26. $\frac{2^3}{2}$
27. $\frac{a^6}{a^4}$
28. $\frac{3^4}{3}$
29. $\frac{10^5}{10^3}$
30. $\frac{2^8}{2^4}$
31. $\frac{10^4}{10}$
32. $\frac{3^2}{2^2}$
33. $\frac{10^3}{10^1}$
34. $\frac{10^5}{10^2}$
35. $\frac{4^7}{4^5}$

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36. $a^{3x} \div a^x$
37. $a^3 \div a^x$
38. $b^{x-1} \div b^2$
39. $3^y \div 3^y$
40. $a^x \div a^{x-2}$
41. $x^{n+2} \div x^{n+1}$
42. $c^{n-3} \div c^2$
43. $3^a \div 3^b$
44. $5^n \div 5$

A table such as the one below is useful in multiplication and division.

$2^1 = 2$	$2^4 = 16$	$2^7 = 128$	$2^{10} = 1024$
$2^2 = 4$	$2^5 = 32$	$2^8 = 256$	$2^{11} = 2048$
$2^3 = 8$	$2^6 = 64$	$2^9 = 512$	$2^{12} = 4096$

EXAMPLE 1. Using the table, multiply 128 by 16.

$$128 = 2^7,\ 16 = 2^4;\ \ 2^7 \cdot 2^4 = 2^{11} = 2048$$

Hence, $128 \times 16 = 2048$. Check your answer by multiplication.

EXAMPLE 2. Divide 4096 by 256.

$$4096 = 2^{12},\ 256 = 2^8,\ 2^{12} \div 2^8 = 2^4 = 16$$

Hence, $4096 \div 256 = 16$. Check this answer by division.

Using the table:

(a) Multiply 256 by 8; 32 by 128; 512 by 8.

(b) Divide 2048 by 256; 4096 by 128; 4096 by 32.

Dividing a Monomial by a Monomial

You have learned that $\frac{ab}{c} = a\left(\frac{b}{c}\right)$ or $\left(\frac{a}{c}\right)b$. $\left(\text{Try this with } \frac{6 \times 4}{2}.\right)$ This means that any factor in the numerator of a fraction may be divided by any factor in the denominator. For example, to divide $6\,a^3b^2$ by $3\,a^2b$, write it as $\frac{6\,a^3b^2}{3\,a^2b}$, then divide 6 by 3, a^3 by a^2, and b^2 by b. Thus, $\frac{6\,a^3b^2}{3\,a^2b} = 2\,ab$.

(1) Are the following examples correct?

(a) $\dfrac{6\,a^3b^2}{2} = 3\,a^3b^2$

(b) $\dfrac{6\,a^3b^2}{b} = 6\,a^3b$

(c) $\dfrac{6\,a^3b^2}{a^2} = 6\,ab^2$

(d) $\dfrac{15\,x^3y^2z}{5\,xy^2} = 5\,x^2z$

Remember that any example in division may be checked by multiplying the divisor by the quotient. The result should be the dividend.

Exercises

Divide as indicated:

1. $10x \div 2$
2. $10x \div x$
3. $x^4 \div x^3$
4. $5x^4 \div x^3$
5. $15a^2 \div 5$
6. $15a^2 \div a$
7. $5xy \div 5$
8. $5xy \div x$
9. $5xy \div 5x$
10. $5xy \div xy$
11. $10x^4 \div 2x$
12. $-15x^4 \div 3x$
13. $20y^5 \div 4y^3$
14. $-20p^5 \div 5p$
15. $-12n^7 \div 4n^7$
16. $18x^5 \div 3x$
17. $-16x^4 \div x^2$
18. $-16x^4 \div -4x^3$
19. $24a^3 \div 6a^3$
20. $8a^2b \div 2a$
21. $8a^2b \div ab$
22. $8a^2b \div 2a^2$
23. $8a^2b \div 2a^2b$
24. $(-5a^6) \div -1$
25. $x^5y^4 \div x^5$
26. $-5n^5 \div 5$
27. $54p^3 \div -6p$
28. $a^2b^2 \div a^2b^2$
29. $m^3n^2 \div mn$
30. $24s^4t^3 \div (-3s^2t)$
31. $\dfrac{-32p^3}{4p^2}$
32. $\dfrac{a^2b^3}{b^3}$
33. $\dfrac{-6a^5}{2a^2}$
34. $\dfrac{15x^4y^2}{-3x^3y^2}$
35. $\dfrac{-18x^6y^2}{6x^6y}$
36. $\dfrac{-4a^4b^3c}{-2a^3b^3}$
37. $\dfrac{4x^2y^3}{9xy^2}$
38. $\dfrac{20x^4y^5}{-2xy^5}$
39. $\dfrac{(5x)^2}{5}$
40. $\dfrac{2a+2b}{2}$
41. $\dfrac{a^2+ab}{a}$
42. $\dfrac{35x^2+14xy}{-7x}$

Perform the indicated operations:

43. $\dfrac{4ab(7a^2b^3)}{-14a^3b^3}$
44. $\dfrac{4ab^2(-5ab^3)}{10a^2b^2}$
45. $\dfrac{(-3ab^2)^2}{6ab}$
46. $\dfrac{(3ab^2)^2(-6a^3b)}{(3a^2b^2)^2}$
47. $\dfrac{(5ab)^2(-20a^3b)}{4a^2b^3}$
48. $\dfrac{(a^{n+1})^2a^{2n}}{a^{4n+2}}$

Dividing a Polynomial by a Monomial

Use several values of a and b to check the correctness of the following statement: $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$. This principle gives us the rule for dividing a polynomial by a monomial.

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial and add the results.

(1) Find two mistakes in the following. Tell what is wrong.

(a) $\frac{3a + 6b}{3} = a + 2b$ (c) $\frac{x^3 + x^2 + x}{x} = x^2 + x$

(b) $\frac{2a + b}{2} = a + b$ (d) $\frac{ax + bx}{x} = a + b$

Check division by multiplying the divisor by the quotient. The result should be the dividend.

Thus, $\frac{x^2 + x}{x} = x + 1$ if $x(x + 1) = x^2 + x$.

In the following set of examples you will find division of monomials by monomials as well as polynomials by monomials. Be sure to distinguish between the two kinds of examples.

Exercises

Divide as indicated:

1. $(2a + 6b) \div 2$
2. $(2a - 6b) \div 2$
3. $(ax + ay) \div a$
4. $(abc + axy) \div a$
5. $6a \div 2$
6. $6a \div a$
7. $(bx^2 + by^2) \div b$
8. $(-bx + by) \div b$
9. $6a \div -2$
10. $3ab \div b$
11. $(5b + bx) \div b$
12. $(6x - 12) \div 6$
13. $\frac{3a + 6}{3}$
14. $\frac{3a + 3}{3}$
15. $\frac{8a - 4b}{2}$
16. $\frac{3ab}{-a}$
17. $\frac{-3ab}{-3}$

18. $\frac{-3ab}{b}$

19. $\frac{9a - 4ab}{a}$

20. $\frac{8a - 4b}{2}$

21. $\frac{15x + 10y}{5}$

22. $\frac{15x - 10y}{5}$

23. $\frac{9a - 3b + 6c}{3}$

24. $\frac{9a - 3b + 6c}{-3}$

25. $\frac{7x + 14y - 21}{7}$

26. $\frac{8x - 6y - 2}{2}$

27. $\frac{-x^2 + 6x - 8}{-1}$

28. $\frac{4x^2 - 2x}{2}$

29. $\frac{4x^2 - 2x}{2x}$

30. $\frac{18a^3 + 6a^4}{-3a}$

31. $\frac{2xy - 3xz}{-x}$

32. $\frac{x^3 + x^2 - x}{x}$

33. $\frac{a^2 - a}{a}$

34. $\frac{a^2 + 2a}{a}$

35. $\frac{4x^2 + 2x + 2}{2}$

36. $\frac{\pi R^2 - \pi r^2}{\pi}$

37. $\frac{-a^2 + 2ab - b^2}{-1}$

38. $\frac{-3ab}{b}$

39. $\frac{2b^2}{b}$

40. $\frac{a^2b^3 - a^3b^2}{ab}$

41. $\frac{a - 2a^2 + 3a^3}{a}$

42. $\frac{-4a^2 + 8a - 12}{-4}$

43. $\frac{x^4y^3 - x^2y^2 + xy}{-xy}$

44. $\frac{a^2x^3y - 5ax^3y^2}{ax^2y}$

45. $\frac{2.8a^2b - 2.1ab^2}{7a}$

46. $\frac{x^3 - 4x^2 + 3x + 2}{0.1}$

47. $\frac{15a^2b^2 - 10a^2b^3 + 25a^3b^2}{-5a^2b^2}$

48. $\frac{39x^3yz^2 - 65xy^2z^3 - 78x^2y^2z^2}{13xyz^2}$

Dividing a Polynomial by a Polynomial

Long division in algebra, like long division in arithmetic, will cause you no difficulty if you remember that it is really a series of subtractions.

To divide 17 by 5, you could subtract 5 from 17, then subtract 5 from the remainder, and so on until the remainder is less than 5. In this case, you subtract 5 three times and the remainder is 2. Hence $17 \div 5 = 3\frac{2}{5}$.

$$\begin{array}{r} 17 \\ \underline{5} \\ 12 \\ \underline{5} \\ 7 \\ \underline{5} \\ 2 \end{array}$$

In this example, you subtracted the divisor only once on each subtraction. With larger numbers this procedure would take too long so that the divisor is subtracted several times on each subtraction. For example, let us divide 3924 by 27. You say, "27 goes into 39 once." This really means that "27 goes into 3924 one hundred times." You then subtract one hundred times 27 from 3294, as shown at the right. The result is the same as if you subtracted 27 separately one hundred times from 3924. You now have 1224 left of your original number and you keep on subtracting. "27 goes into 122 four times;" that is, "27 goes into 1224 forty times." You subtract 40×27 or 1080 from 1224 and have 144 left. "27 goes into 144 five times" and you subtract 5×27 or 135 from 144. The remainder is 9.

$$\begin{array}{r|r} & 100 \\ 27 & 3924 \\ & \underline{2700} \\ & 1224 \end{array} \qquad \begin{array}{r|r} & 40 \\ 27 & 1224 \\ & \underline{1080} \\ & 144 \end{array} \qquad \begin{array}{r|r} & 5 \\ 27 & 144 \\ & \underline{135} \\ & 9 \end{array}$$

Putting these three steps together, your work looks like this. In all you have subtracted 27, $100 + 40 + 5$ times and the remainder is 9. Hence $3924 \div 27 = 145\frac{9}{27}$ or $145\frac{1}{3}$. Ordinarily you do not put in the zeros as you divide and you do not "bring down" numbers until you need them. They are put in here to make it plainer to you that long division is a series of subtractions. You keep on subtracting until the remainder is less than the divisor.

$$\begin{array}{r|r} & 5 \\ & 40 \\ & 100 \\ 27 & 3924 \\ & \underline{2700} \\ & 1224 \\ & \underline{1080} \\ & 144 \\ & \underline{135} \\ & 9 \end{array}$$

You should now be able to understand the method of dividing a polynomial by a polynomial. Like long division in arithmetic, it consists mainly of a succession of subtractions.

EXAMPLE 1. Divide $6x^3 + x^2 - 12x + 8x$ by $2x - 1$

(1) Put your work in the following form:

$$2x - 1 \,\overline{)\, 6x^3 + x^2 - 12x + 8}$$

(2) Divide the first term of the dividend by the first term of the divisor: $6x^3 \div 2x = 3x^2$. This tells you how many times to *subtract* $2x - 1$ from the dividend. Find $3x^2$ times $2x - 1$ (It is $6x^3 - 3x^2$.) and subtract. The remainder is $4x^2 - 12x + 8$. Put $3x^2$ as the first term of the quotient.

$$\begin{array}{r|l}
 & 3x^2 \\
\hline
2x - 1 & 6x^3 + x^2 - 12x + 8 \\
 & 6x^3 - 3x^2 \\
\cline{2-2}
 & \qquad 4x^2 - 12x + 8
\end{array}$$

(3) Now consider the remainder $4x^2 - 12x + 8$ as a new dividend and proceed as before. Divide the first term of the remainder by the first term of the divisor: $4x^2 \div 2x = 2x$. Subtract $2x$ times $2x - 1$ (It is $4x^2 - 2x$.) from the remainder; this gives $-10x + 8$ as the new remainder. Put $+2x$ as the second term of the quotient.

$$\begin{array}{r|l}
 & 3x^2 + 2x \\
\hline
2x - 1 & 6x^3 + x^2 - 12x + 8 \\
 & 6x^3 - 3x^2 \\
\cline{2-2}
 & \qquad 4x^2 - 12x + 8 \\
 & \qquad 4x^2 - 2x \\
\cline{2-2}
 & \qquad\qquad -10x + 8
\end{array}$$

The remainder, $-10x + 8$, is what you have left of the original polynomial after subtracting the divisor $(3x^2 + 2x)$ times.

(4) Proceed as before. $-10x \div 2x = -5$. Subtract -5 times $2x - 1$ from the remainder. Put -5 as the third term of the quotient.

$$\begin{array}{r|l}
 & 3x^2 + 2x - 5 \\
\hline
2x - 1 & 6x^3 + x^2 - 12x + 8 \\
 & 6x^3 - 3x^2 \\
\cline{2-2}
 & \qquad 4x^2 - 12x + 8 \\
 & \qquad 4x^2 - 2x \\
\cline{2-2}
 & \qquad\qquad -10x + 8 \\
 & \qquad\qquad -10x + 5 \\
\cline{2-2}
 & \qquad\qquad\qquad\quad 3
\end{array}$$

(5) Since the remainder, 3, no longer contains the first term of the divisor, you know that the division is completed. The answer is $3x^2 + 2x - 5 + \frac{3}{2x-1}$; that is, the divisor has been subtracted from the dividend $(3x^2 + 2x - 5)$ times with a remainder of 3.

To check, multiply $3x^2 + 2x - 5$ by $2x - 1$ and then add 3 to the result.

Note that while in arithmetic you can write a remainder next to the quotient as $3\frac{2}{5}$, without any sign between, in algebra the plus sign is necessary. Why?

EXAMPLE 2. Divide $-7x - 20 + 6x^2$ by $4 + 3x$.

We first arrange the terms of the dividend and the divisor according to some regular order, in this case according to the descending powers of x.

$$\begin{array}{r|l} & 2x - 5 \\ \hline 3x + 4 & 6x^2 - 7x - 20 \\ & 6x^2 + 8x \\ \hline & -15x - 20 \\ & -15x - 20 \\ \hline \end{array}$$

(1) How do we get the $2x$?
(2) How do we get $6x^2 + 8x$?
(3) How do we get $-15x - 20$?
(4) How do we obtain -5?

Check this solution by multiplication.

Divide $6x^2 - 7x - 20$ by $2x - 5$. Compare this exercise with the preceding example.

Exercises

Divide as indicated:

1. $a^2 + 6a + 8$ by $a + 4$
2. $b^2 + 8b + 15$ by $b + 3$
3. $c^2 - 3c - 28$ by $c - 7$
4. $n^2 - 12n - 28$ by $n + 2$
5. $x^2 + xy - 2y^2$ by $x - y$
6. $6x^2 - 28x + 30$ by $2x - 6$
7. $10b^2 - 11b + 3$ by $2b - 1$
8. $x^2 - 12x + 40$ by $x - 6$
9. $6a^2 - 13a + 7$ by $2a - 3$

10. $c^2 - c - 20$ by $c + 4$

11. $p^2 - p - 30$ by $p + 5$

12. $a^2 - 6\,ab + 9\,b^2$ by $a - 3\,b$

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13. $8\,a^2b + 7\,ab^2 + 3\,a^3 + 5\,b^3$ by $3\,a + 2\,b$

14. $6\,a^2 - 12 - 2\,a + a^3$ by $a^2 - 2$

In the dividend of Ex. 21 below there is no x^2 term. Write the dividend as $x^3 + 0\,x^2 + 6\,x + 4$ and proceed as usual. Whenever a term is missing from the dividend, you should write it in with a zero coefficient.

15. $x^3 + 6\,x + 4$ by $x + 2$

16. $x^3 + 4\,x - 3$ by $x + 5$

17. $x^3 - 6\,x^2 + 1$ by $x + 2$

18. $5\,x + 6\,x^2 - 4$ by $2\,x - 1$

19. $4 - 6\,x^2 + 3\,x$ by $x + 5$

20. $x^3 - 8$ by $x - 2$

21. $p^3 - 125$ by $p - 5$

22. $8\,x^3 - 1$ by $2\,x - 1$

Meaning of Literal Numbers[1]

In the last three chapters you have for the most part been working with letters without substituting arithmetic numbers for them. In this kind of work it is easy to lose sight of the fact that letters in algebra are numbers. If you are to continue your study with real understanding, you must remember this fact, and also that the letters sometimes represent "any number" and sometimes "some number" or "some numbers." The exercises that follow in which the letters are used in equations should help you to a fuller appreciation of these facts. Carry out the exercises in the following manner:

1. *State what each equation says in words.*

For example, the equation in Ex. 3 says in words: "If I have two numbers, add them, subtract the second from the first, and then multiply the one result by the other, I will get the same number as if I squared both numbers and then subtracted the second square from the first square." Reading the statements in words should help you to remember not only that the letters are numbers, but should show you how much more

[1] To the Teacher. See Note 17 on page 461.

briefly and simply general statements about numbers can be made by using algebraic symbols.

2. *Choose numerical values for the letters, substitute them in the equations, and decide in which equations the letters may be "any numbers."*

If you find that you can substitute any value you please for the letters and the equation still remains true, you have the situation in which the letters represent "any numbers."

If you find that the algebraic statements are true for only a few numbers or for only one number, you have the case where the letters represent "some numbers" or "some number."

3. *Perform the operations indicated on the left side of each equation to see if you get what is on the right side in order to tell in which equations the letters may be "any numbers."*

In 2. above, substituting only a few numbers is not sufficient to show that an equation is true for *all* numbers. It might be true for just a few numbers and not for others. You can readily see that it might take a long time by this method to determine whether or not an algebraic statement is true for all numbers. The procedure of (3) will give you a more practical method of doing this.

If you perform the operations indicated on the left side of an equation and get what is on the right side of the equation, you know that the algebraic statement is true in general and that the letters may be "any numbers." If the left side of the equation does not turn out to be the same as the right side, you know that the statement is not true for all numbers. Some of these equations may be untrue in general unless some or all of the letters are zero while others may be true only for "some numbers" or "some one number."

CAUTION. Some statements are true for all values of the letters except a special few. For example, $\frac{a^2 - b^2}{a - b} = a + b$ unless $a = b$. If $a = b$, both the numerator and the denominator are zero and the value of the left side of the equation cannot be determined.

Exercises

1. $7a + 2a - 5a = 4a$
2. $3(a + b) = 3a + b$
3. $(a + b)(a - b) = a^2 - b^2$
4. $a(b - c) = ab - ac$
5. $a - (b + c) = a - b + c$
6. $a - (b - c) = a - b + c$
7. $a + b(c + d) = a + bc + bd$
8. $(a + b)(c + d) = ac + bc + ad + bd$
9. $(n + 4)(n + 3) = n^2 + 4n + 18$

(NOTE ON EX. 9. If you tried $n = 2$ in this equation, without further investigation, you might be led to believe it is true for any number. If, however, you try any other number you will find it is not true. When you multiply $n + 4$ by $n + 3$, you will show in another way that the statement is not true in general. This is a case where the letter means "some number." It is true only for $n = 2$.)

10. $\frac{a + b}{3} = \frac{a}{3} + b$
11. $\frac{1}{2}ab = (\frac{1}{2}a)(\frac{1}{2}b)$
12. $\frac{a + b - c}{3} = \frac{a}{3} + \frac{b}{3} - \frac{c}{3}$
13. $(a + b)^2 = a^2 + 2ab + b^2$
14. $(n + 5)(2n - 3) = 2n^2 + 7n - 15$
15. $\frac{n^2 + 3n - 5}{n - 2} = n + 5 + \frac{5}{n - 2}$

(Is this statement true for $n = 2$?)

16. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$
17. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$

Chapter Summary

This chapter has broadened your knowledge of multiplication and division of algebraic numbers by showing you how these processes work when you are dealing with exponents, monomials, powers of monomials, and various combinations of monomials, binomials, and polynomials.

You should review the rules developed for multiplying and dividing these expressions and go back and do at least one typical example in each set of exercises.

Chapter Review

1. Read each of the following in two ways: n^2, n^3, ab^2, a^3b^2.

2. Does n^3 mean $n + n + n$ or $n \times n \times n$?

3. Does $3\,n$ mean 3 times n, $n + n + n$, or $n \times n \times n$?

4. Does $5\,b^3$ mean 5 times $b + b + b$, $5\,b \times 5\,b \times 5\,b$, or $5 \times b \times b \times b$?

5. Does $(5\,b)^3$ mean $5\,b \times 5\,b \times 5\,b$, or $5 \times b \times b \times b$?

6. Explain by using the meanings of x^5 and x^3 why it is that $x^5 \cdot x^3$ is x^8. Also why $x^5 \div x^3$ is x^2.

7. Does $2^2 \times 2^3 = 4^5$? Explain.

State each of the following in words:

8. $x^a \cdot x^b = x^{a+b}$

9. $x^a \div x^b = x^{a-b}$

10. $a \times 0 = 0$

11. $a \div 0$ is impossible

Multiply as indicated:

12. $a^3 \times a^4$

13. $-\,a^2 \times a^3$

14. $-\,a^5 \times -\,a^2$

15. $b^5 \times b^3 \times b^2$

16. $2^3 \times 2^4$

17. $10^{2.4} \times 10^{3.6}$

Divide as indicated:

18. $\dfrac{x^8}{x^5}$

19. $\dfrac{a^3}{a^2}$

20. $\dfrac{5^4}{5}$

21. $\dfrac{3^7}{3^6}$

22. $\dfrac{-\,n^5}{-\,n^2}$

23. $\dfrac{b^{10}}{b^2}$

24. What are the steps in multiplying $2\,a^3b^2c$ by $3\,ab^2$? What principles of multiplication are involved? (See page 195.)

Multiply as indicated:

25. $(+\,2)(7\,b)$

26. $(+\,b)(7\,b)$

27. $(3\,a^2)(b)$

28. $(3\,a^2b)^2$

29. $(-\,5\,ab^3)^3$

30. $(5\,x^2y)(-\,3\,xy^2)(x^3)$

31. $(-\,7\,x^3y^2z)(-\,4\,xy^3z^5)$

32. $(3\,a^2)^2(-\,ab)(9\,ab^3c^2)$

33. State in words: $a(b + c + d) = ab + ac + ad$. What is the rule for multiplying a polynomial by a monomial?

Perform the indicated operations:

34. $5(2a + 3b)$
35. $-5(2a - 3b)$
36. $-5(-2a - 3b)$
37. $-3a(a^2 - b^2)$
38. $2a(a^2b)$
39. $a(a^2 - 2ab + b^2)$
40. $x(x^3 - x^2 + x - 1)$
41. $2m^2n(5m^2 - 3n)$
42. $3x + 2(6x - 5)$
43. $3x - 2(6x - 5)$
44. $-3(3x - 4y) + 5(4x - 3y)$
45. $x^2(x^2 - 2x) - x^3(x - 1)$

Perform the indicated operations:

46. $(a + b)(c + d)$
47. $(a - 2b)(2a + 3b)$
48. $(3x - 4)(3x + 4)$
49. $(5a - 2b)^2$
50. $(a + b)^3$
51. $3x - 4(2x + 3)$
52. $(3x - 4)(2x + 3)$
53. $(x^2 - y^2)(x + y)$
54. $(a^2 - 2b^2)(3a^2 + 5b^2)$
55. $(x^2 + 2x + 1)(2x - 3)$
56. $7n - (3n + 2)(3n - 2)$
57. $(2x - 3)(3x - 2) - 5(x + 2)^2$
58. Does $(a - b)^2 = (b - a)^2$? How do you account for your answer?

Divide as indicated:

59. $8a \div 2$
60. $8a \div a$
61. $8a \div 4a$
62. $24a^2b \div a$
63. $24a^2b \div ab$
64. $24a^2b \div 6a^2$
65. $-24a^3b^2c \div a^2bc$
66. $-24a^3b^2c \div -6b^2c$
67. $24a^3b^2c \div -6abc$
68. $\dfrac{6a - 12b}{3}$
69. $\dfrac{a^3 + a^2 + a}{a}$
70. $\dfrac{a^2b^3c + ab^2c^3}{ab^2c}$

CUMULATIVE REVIEW[1]

The purpose of the cumulative reviews in this book is to keep you refreshed on the various processes that have been used and to cause you to go over again the ideas that have been presented in the course. As you work the exercises, turn back to the text and review anything on which you are hazy. We make newly learned things a part of our permanent mental capital only by fixing them firmly in memory and reviewing their relationships in the larger way that our advancing knowledge makes possible.

Evaluate each of these expressions for $a = 3$ *and* $b = -2$. (*Use* $\pi = \frac{22}{7}$.)

1. $\frac{1}{2}ab$

2. $\frac{a+b}{3}$

3. $a + \frac{b}{3}$

4. $\frac{a}{3} + b$

5. $\frac{2ab}{3}$

6. $\frac{\pi a}{b}$

7. $\frac{\pi a^3}{3}$

8. $\pi a(a + b)$

9. $\pi a^2 b$

10. $\pi(a - b)^2$

Find the value of each of the following when a is 5 and b is -4:

11. $2ab$

12. $2a + b$

13. $a + 2b$

14. $a^2 + b^2$

15. $(a + b)^2$

16. $2a^2b$

17. $2ab^2$

18. $a(a + b)$

19. $a - (a - b)$

20. $5b - 7$

21. $\frac{1}{2}(3a - 2b)$

22. $\frac{a+b}{5}$

23. $\frac{a+b}{2a-b}$

Find the value of each of the following for the values given:

24. $a^2 + 2ab$; $a = -7$, $b = 3$

25. $x + 3(x + 2)$; $x = 9$

26. $(x + 3)(x + 2)$; $x = 9$

27. $\frac{2a + 3b}{5}$; $a = 2$, $b = 9$

28. $\frac{b}{b^2 + c^2}$; $b = 3$, $c = -1$

[1] To the Teacher. See Note 18 on page 461.

29. What is the value of $3^2 - 2^3 + 3(2^2) - 1^3$?

30. Match the word phrases at the left with the formulas at the right. Then cover the right-hand column and write the formulas which correspond to the word phrases.

(*a*) Area of a rectangle	1. $A = \frac{1}{2}bh$
(*b*) Volume of a cylinder	2. $p = 3a$
(*c*) Perimeter of a triangle	3. $V = e^3$
(*d*) Distance formula	4. $A = \pi r^2$
(*e*) Area of a circle	5. $p = 2(l + w)$
(*f*) Perimeter of an equilateral triangle	6. $p = a + b + c$
(*g*) Area of a parallelogram	7. $V = lwh$
(*h*) Perimeter of a rectangle	8. $d = rt$
(*i*) Circumference of a circle	9. $A = lw$
(*j*) Perimeter of a square	10. $p = 4s$
(*k*) Volume of a cube	11. $A = bh$
(*l*) Area of a triangle	12. $A = s^2$
(*m*) Volume of a rectangular solid	13. $i = prt$
(*n*) Simple interest formula	14. $V = \pi r^2 h$
(*o*) Area of a square	15. $c = 2\pi r$

31. What is the perimeter of a triangle whose sides are $5\frac{1}{2}$ ft., $3\frac{3}{4}$ ft., and $3\frac{1}{4}$ ft.?

32. What is the perimeter of a rectangle whose length is 6.7 ft. and whose width is 4.3 ft.?

33. Find the area of a rectangle whose length is 6.7 ft. and whose width is 4.3 ft.

34. What is the area of a rectangle whose length is 8.5 ft. and whose width is 4.3 ft.?

35. What is the area of a triangle whose base is 124 in. and whose height is 41 in.?

36. What is the area of a triangle whose base is $12\frac{1}{2}$ ft. and whose altitude is $7\frac{3}{4}$ ft.?

37. How many cubic yards of earth will it take to fill a rectangular ditch 15 ft. long, 2 ft. wide, and 3 ft. deep?

38. What is the volume of a cube whose edge is 2.5 in.? Give the answer to the nearest tenth.

39. Which of these formulas states the rule for finding the interest on any principal for one year at 6%?

(*a*) $i = \frac{6p}{100}$ (*b*) $i = .6p$ (*c*) $i = .06p$ (*d*) $i = \frac{6}{100}p$

40. What is the value of i in the formula $i = prt$, when $p = 150$, $r = .045$, and $t = 3$?

41. What is the interest on $460 for 2 years at 6%?

42. How long will it take $1000 to earn $500 interest when it is invested at 6% simple interest?

43. Write as a formula: The total surface S of a cylinder is equal to $2\pi r$ times the sum of the radius r and the height h.

44. Write a formula for changing f ft. and n in. to inches.

45. Write a formula for changing p lb. and n oz. to ounces.

46. Make a formula for the number of gallons of water that may be placed in a rectangular tank l ft. by w ft. by h ft. (1 cu. ft. = $7\frac{1}{2}$ gal.)

47. A certain club asks an initiation fee of 50 cents and dues of 10 cents a month. Write a formula for finding C, the cost in dollars of membership in this club for any number of months n.

48. How many cents are there in a quarters, b dimes, and c nickels?

49. If Madge earned $4 in n hours, what did she earn in 1 hour, assuming that she earned the same amount each hour?

50. How far will a train travel in 1 hour if it travels m miles in h hours?

51. If the perimeter of a square is n ft., what is the length of each side?

52. If the area of a square is n sq. ft., what is the length of each side?

53. How many yards are there in x ft.?

54. The price in dollars of a sweater is c. At a discount of 25% what will it cost?

55. How many factors has $3\,abc$? If you wished to multiply this expression by 6, how many factors would you multiply by 6?

56. How many terms has $3\,a + 2\,b + 5\,c$? If you wished to multiply this expression by 6, how many of the terms would you multiply by 6?

57. In order to multiply an indicated product by 3, multiply __?__ factor(s) by 3.

58. In order to multiply an indicated sum by three, multiply __?__ addend(s) by 3.

59. The value of $6\,x - 10$ depends upon __?__.

60. The distance Fred can ride on his bicycle at an average rate of 5 miles an hour depends upon the __?__ he spends riding.

61. From a study of the temperature graph at the bottom of the page answer the questions below.

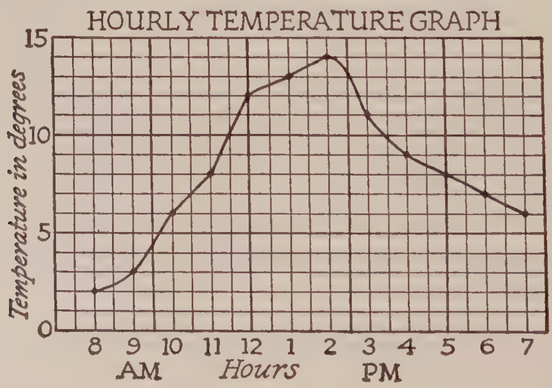
(*a*) What was the temperature at 9 o'clock? at 11 o'clock? at 4 o'clock?

(*b*) At what hours was the temperature 8°?

(*c*) When was the temperature the highest?

(*d*) Does temperature depend upon the time of day only?

(*e*) Would this graph serve as a picture of temperature conditions every day of the year? for a particular date every year?



62. The area of a square depends upon the _?_ of a _?_.

63. The volume of a cylinder depends upon its _?_ and upon the _?_ of the circular base.

64. The formula $A = \pi r^2$ shows that the area of a _?_ depends upon its _?_.

65. If $a = b - c$ and c increases, a _?_.

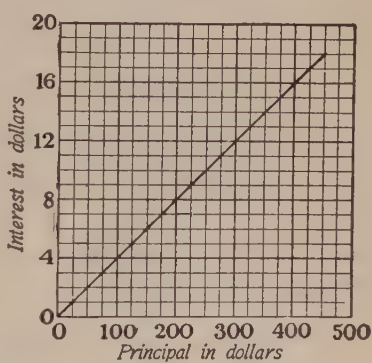
66. Does the value of $2a + 3b - 5c$ increase or decrease when c increases?

67. The number of yards of wallpaper border needed to go around a rectangular room depends upon the _?_ and _?_ of the room.

68. The graph at the right, below, shows the interest for one year on the number of dollars invested at 4%.

(*a*) What does the horizontal scale on this graph show?

(*b*) Complete the following sentence by supplying the missing word: One large space on the vertical scale represents _?_ dollars.



(*c*) Use the graph to find how much interest you would get per year on $300; on $250; on $200; on $150; on $350; on $50.

(*d*) Use the graph to tell how much a man would need to invest at 4% to get $8 interest per year; $10 interest; $6 interest; $2 interest; $18 interest.

(*e*) Use the interest graph to complete the following table. Copy the table. Do not write in the book.

Principal	$100	125	175	225	325	375
Interest at 4% . .	$4					

69. Make a table which contains pairs of corresponding numbers for the formula $A = h + 10$ when h is 2, 4, 6, 8, 10.

70. Copy and complete the table, using the formula

$$p = 2a + 3.$$

a	0	1	2	3	4	5	6
p		5			11		

71. Make a table for the formula $F = \frac{9}{5}C + 32$, using the values $C = 0, 10, 20, 30, 60, 80, 100$.

72. Write an equation from which each one of the following tables could be obtained. Make a graph for each.

(*a*)

x	0	1	2	3	4
y	5	7	9	11	13

(*b*)

x	1	2	3	4	5
y	2	5	8	11	14

(*c*)

x	0	1	2	3	4
y	0	1	4	9	16

(*d*)

x	0	1	2	3	4
y	1	2	5	10	17

73. Make a graph of the formula $A = 10h$.

74. Make a graph of the formula $p = 3a + 2$. Using the graph, find the value of p when $a = 2.5$ and the value of a when $p = 4$.

75. Doubling each side of an equilateral triangle multiplies the perimeter by _?_.

76. Doubling each side of a square multiplies the area by _?_.

77. The formula $F = \frac{9}{5}C + 32$ is used to change centigrade degrees to Fahrenheit degrees. Does F increase or decrease when C increases? Is F trebled when C is trebled?

78. If the length of a rectangle is 4 inches more than the side s of a square, and the width is 3 inches less than the side of the square, what is the area A of the rectangle in terms of s? its perimeter p in terms of s?

79. Write a formula for T, the total cost in dollars of n pounds of butter at c cents a pound and p dozen eggs at d cents a dozen.

80. Copy and fill in the following table:

n	1	2	3	4	5	6	7	8	9	10
$5n$	5	10	15							
n^2	1	4	9							

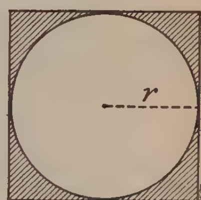
(*a*) What is the difference between any two successive numbers in the first row? Are the differences constant or variable?

(*b*) What is the difference between any two successive numbers in the second row? Are the differences constant or variable?

(*c*) In the third row are the differences constant or variable?

81. What is the simple interest on \$500 at $r\%$ for 3 years?

82. What is the area of the circle shown here? of the square? of the shaded surface between the circle and the square? (Note that the radius of the circle is r.)



83. If $8a^2b^3c^4$ is the dividend and $2a^2bc$ is the divisor, what is the quotient?

84. Divide $x^2 - 2x - 24$ by $x - 2$. Check your answer.

85. Divide $a^3 - a^2b - 10ab^2 - 6b^3$ by $a - 4b$. Check your answer.

86. Divide $4 - 7x^2 + 7x^3 - 4x$ by $x - 1$. Check.

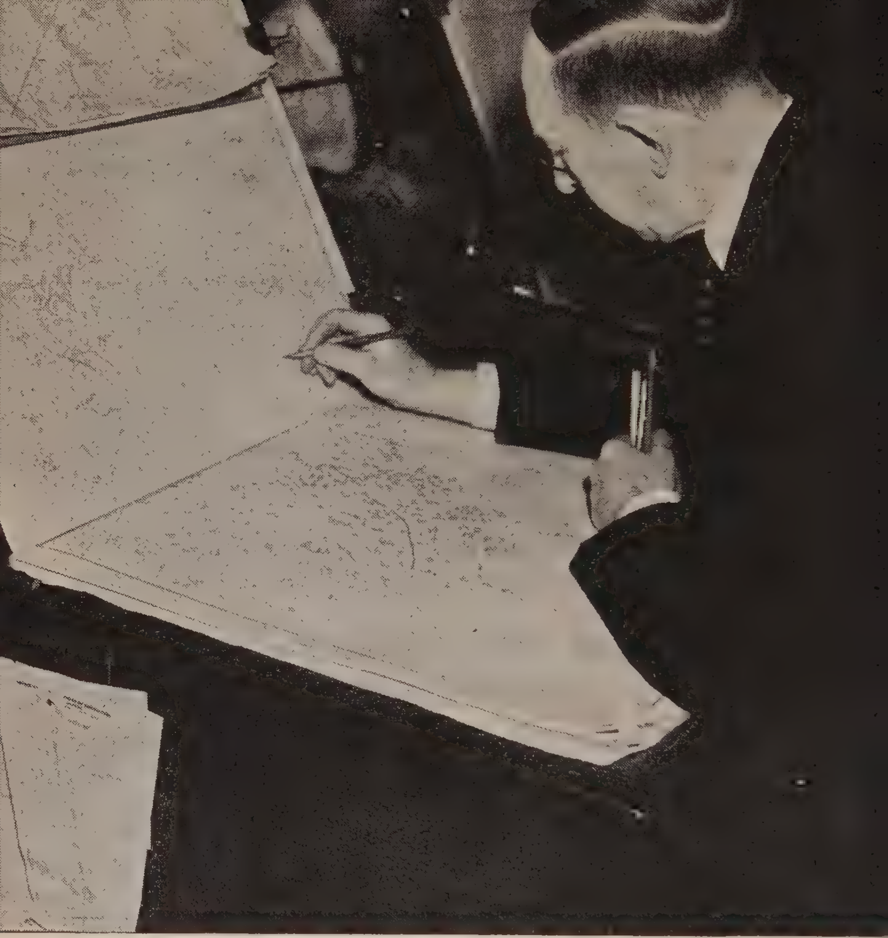
87. Divide $a^3 - 8b^3$ by $a - b$. Check your answer.

Solve and check:

88. $7x - 3(x - 5) = 31$

89. $5(2x + 5) - 2(3x - 2) = 5$

90. $(6x - 4)(2x + 3) - (4x - 3)(3x + 2) = -28$



U. S. Weather Bure

Forecasting the weather is no longer a matter of proverbs or local signs. It is the science of meteorology with mathematics even more advanced than algebra behind it. For aviation it must be as exact as possible. Take up a book on advanced meteorology and you will find that in solving problems connected with the weather, as in the solution of problems in all other fields of science, equations are constantly employed.

Chapter IX

EQUATIONS

The human mind unaided is sufficient for thinking through quantitative problems when they are simple enough. For example, if a merchant buys a suit of clothes for \$28 and wishes to sell it at a price that is a 25% advance over the buying price, he can determine the selling price at once. On the other hand, in this world of complex business, of machines and new ways of communication, of research into the nature of matter and life, quantitative relationships are often far more difficult. In handling them, the world has become dependent upon mathematics beyond arithmetic.

For solving these quantitative problems of modern life, the equation is the master tool. With its algebraic symbols and definite methods of solution it is a powerful aid for the use of number.

Some Facts about Equations

Some equations are satisfied by any values of the letters. Others are satisfied by only particular values. Equations which are satisfied by all values of the letters are known as ***identical equations*** or ***identities***. Equations which are true for only particular values of the letters are called ***conditional equations*** or simply ***equations***.

If an equation has only one unknown number and this number occurs only with the exponent 1 and does not occur in the denominator of any fraction, it is called an ***equation of the first degree in one unknown***.

You have already learned that an equation of the first degree in one unknown ***is satisfied*** when you have found the value of the unknown which makes the two sides of the equation equal and that this value is called the *root* of the equation.

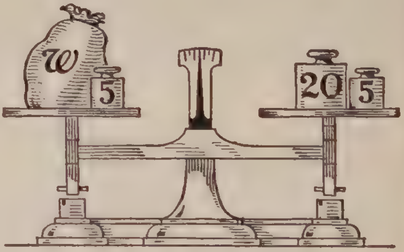
Identical equations may be thought of as declarative sentences. They declare a fact; for example, $(n + 3)^2$ always equals $n^2 + 6n + 9$. Conditional equations may be thought of as interrogative sentences. They ask a question. When you are asked to solve the equation $3n + 4 = 9$, you are asked, "For what value of n is this equation true?"

The equations you have solved so far have had the unknown on one side of the equation only. The other side has consisted of a single number, an arithmetic number. Before you can solve equations with the unknown on both sides, you must analyze more carefully what is done when you solve an equation.

Before you proceed, review pages 34–36 on "Solving Equations." Be sure you understand inverse processes.

Analyzing the Method of Solving Equations †

When you solve the equation $w + 5 = 25$, you say to yourself, "5 has been added to w. To find w, I must subtract 5 from 25." You make the subtraction and get $w = 20$. You subtracted 5 from the right side of the equation. Has it occurred to you that you have also subtracted 5 from the left side of the equation?



An equation is like a balance.

We can illustrate the solution of the equation $w + 5 = 25$ by using the picture at the right, where $(w + 5)$ pounds on the left side balances 25 pounds on the right side. You do not know how many pounds there are in the bag and wish to find out.

If you take the 5-pound weight from the left side, you have only the bag (w pounds) left. Now the scales are out of balance. To restore the balance, you must take 5 pounds from the right side also. The bag now balances the 20-pound weight and you know that it weighs 20 pounds. To find w, you have taken 5 pounds from *both sides*.

Now let us apply this thought to the solution of $w + 5 = 25$. You wish to have w alone on the left side. You have 5 more than you want; so you *subtract* 5 and have w left. (How much is 5 from $w + 5$?) Now your equation is out of balance. To restore the balance, you must subtract 5 from the right side also. The right side then becomes 20 and you have the solution $w = 20$.

In symbols our explanation looks like the following:

SOLUTION.	$w + 5 = 25$
Subtracting 5,	$5 = 5$
RESULT.	$w = 20$

You have subtracted 5 from *both sides* of the equation.

(1) If two numbers are equal, how will the results compare if —

(*a*) the same number is added to both of them?
(*b*) the same number is subtracted from both of them?
(*c*) they are both multiplied by the same number?
(*d*) they are both divided by the same number?

(The results are equal in all these cases.)

(2) What must I do to the left side of the equation $w - 5 = 25$ to have w alone? (5 has been subtracted; so to get w I must add 5 to $w - 5$.)

(3) If I add 5 to $w - 5$, the result is w.

(4) If I add 5 to the left side of the equation, what must I do to the right side to keep the balance? (Add 5 to the right side.)

(5) Write the solution of $w - 5 = 25$, showing what you do.

SOLUTION.	$w - 5 = 25$
Adding 5,	$5 = 5$
RESULT.	$w = 30$

(6) What must you do to the left side of the equation $2n = 10$ to get n alone? (n has been multiplied by 2; so to get n I must divide $2n$ by 2.)

(7) If I divide the left side of the equation by 2, what must I do to the right side to keep the balance? (Divide the right side by 2 also.)

(8) Write the solution of $2n = 10$.

SOLUTION. $2n = 10$

Dividing by 2, $\frac{2n}{2} = \frac{10}{2}$

RESULT. $n = 5$

(9) Analyze the solution of $\frac{n}{2} = 10$ in the same way.

SOLUTION. $\frac{n}{2} = 10$

Multiplying by 2, $\frac{2n}{2} = 2(10)$

RESULT. $n = 20$

The important thing that you have now learned is that equations are solved by operating on *both sides* of the equation. You first decide what you should do to one side, and then you do the same thing to the other side.

The Equation Axioms

The principles you have just used are stated as follows:

If the same number is added to both sides of an equation, the results are equal.

If the same number is subtracted from both sides of an equation, the results are equal.

If both sides of an equation are multiplied by the same number, the results are equal.

If both sides of an equation are divided by the same number, the results are equal. (Division by zero is excepted.)

These are the equation axioms. They may all be summarized thus:

Doing the same thing to both sides of an equation does not destroy the equality.

Exercises

1. I have the equation $5b = 30$. What do I do to both sides of the equation to get $b = 6$?

2. I know that $n + 7 = 21$. What do I do to both sides of the equation to get $n = 14$?

3. I have the equation $\frac{n}{2} = 8$. How do I get from it the equation $n = 16$?

4. How do I get $2n = 10$ from the equation $2n + 1 = 11$?

5. Suppose you know that $n - 4 = 8$. What do you do to get $n = 12$?

6. How do you get $5x = 15$ from the equation $5x - 2 = 13$?

7. If $x^2 + 2x + 3 = 15$, how much is $x^2 + 2x$? How did you get your answer?

8. If $3x^2 - 4x - 7 = 2$, how much is $3x^2 - 4x$? How did you get your answer?

9. If $6x^2 + 4x = 10$, how much is $3x^2 + 2x$? How did you get your answer?

10. If $9x^2 + 6x = 12$, how much is $3x^2 + 2x$? How did you get your answer?

11. If $x^2 + 2x = 1$, how much is $2x^2 + 4x$?

Solve and check the following equations. Be ready to state orally what you did to both sides of each equation in solving it. You should be able to work from right to left as well as from left to right.

12. $x - 7 = 2$
13. $n + 4 = 6$
14. $3x = 9$
15. $\frac{x}{4} = 3$
16. $4y = 7$
17. $7y = 4$
18. $10 = 5x$
19. $1 = y - 7$
20. $\frac{1}{4}x = 9$
21. $x - 6 = 1$
22. $3y = 8$
23. $8y = 3$
24. $11 = x + 10$
25. $a + 8 = 20$
26. $a + 2\frac{1}{2} = 5$
27. $5x = 12$
28. $\frac{1}{5}a = 2$

29. $12x = 5$
30. $2a = 3$
31. $4n = 7$
32. $5n = 3$
33. $7n = 4$
34. $3n = 5$
35. $a + 3.2 = 9.6$
36. $y - 4.7 = 9.3$
37. $y + 4.7 = 9.3$
38. $p - \frac{1}{2} = 4$
39. $p - \frac{2}{3} = 3\frac{1}{3}$
40. $2 = y - 5$
41. $\frac{x}{3} = 11$
42. $x - \frac{1}{2} = \frac{1}{2}$
43. $2 = \frac{n}{9}$
44. $21 = x - 11$
45. $\frac{1}{10}x = 7$
46. $x + 2.5 = 3.7$
47. $\frac{1}{5} = x - \frac{1}{5}$
48. $y + 1.8 = 3.4$
49. $x - 3\frac{1}{4} = 2\frac{3}{4}$
50. $x - 5\frac{1}{8} = \frac{7}{8}$
51. $x - 6.2 = 5.8$

Two-Step Equations

The solution of the equations below are not different from the solutions you have already carried through in previous chapters except in the more careful way that you should think about each step.

(1) Solve $2n + 5 = 9$. The left side consists of two terms. You wish to keep on that side only the term containing n. To do this, you must "get rid" of 5. Subtract 5; because of some of your future work you should think of it as adding -5. What is the result of adding -5 to $2n + 5$? (Result, $2n$.) To restore the balance, you must add -5 to the right side also. Now you have $2n = 4$. To solve this equation, divide *both sides* by 2.

SOLUTION.	$2n + 5 = 9$
Adding -5,	$2n = 4$
Dividing by 2,	$n = 2$
CHECK.	$4 + 5 = 9$

(2) Analyze the solution of $2n - 5 = 9$ in the same way that the preceding solution was handled.

(3) Solve $\frac{2n}{3} = 5$. Here we shall get rid of the 3 and the 2 on the left side by two separate steps. The $2n$ has been divided by 3; hence multiply both sides by 3. How much is $\frac{2n}{3} \times 3$? (It is $2n$.) You now have $2n = 15$.

SOLUTION. $\frac{2n}{3} = 5$

Multiplying by 3, $2n = 15$

Dividing by 2, $n = 7\frac{1}{2}$

CHECK. $\frac{2(7\frac{1}{2})}{3} = \frac{15}{3} = 5$

This equation could have been written $\frac{2}{3}n = 5$. $\frac{2n}{3}$ and $\frac{2}{3}n$ mean the same thing. Check this by using several values of n.

Each side of an equation should be simplified as much as possible before you begin to solve it.

Exercises

Solve and check these equations. Be ready to state what you did to both sides of each equation as you proceeded from step to step.

1. $4x + 5 = 17$
2. $2p - 3 = 9$
3. $7n - 5 = 16$
4. $5b + 7 = 42$
5. $17 = 5 + 3x$
6. $22 = 8 + 7x$
7. $2x + 3 = 14$
8. $7x - 1 = 15$
9. $2x + 3x = 30$
10. $4x + 7x - 2x = 36$
11. $7x - 2x + 3x = 12$
12. $5x + 3 - 2x = 18$
13. $7x - 5 + 2x = 19$
14. $8 + 6x - 5 = 17$
15. $\frac{2x}{5} = 1$
16. $\frac{2}{5}x = 3$
17. $\frac{3x}{2} = 1$
18. $\frac{3x}{4} + 2 = 5$
19. $\frac{2}{3}x - 1 = 4$

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20. $8\frac{1}{2}n + 3\frac{1}{4}n - 5\frac{3}{4}n = 108$
21. $4\frac{2}{3}n - \frac{1}{2}n - 2\frac{5}{6}n = 24$
22. $6\frac{1}{4}x - 5\frac{3}{4}x + 15 = 21$

23. $0.7\,y - 0.1\,y + 0.7 = 1.3$

24. $0.5\,x + 0.7\,x - 0.6 = 3.0$

25. $3\,n - 4\frac{1}{4} = 5\frac{2}{3}$

26. $8\,n + 1\frac{1}{2} = 6\frac{1}{3}$

27. $3\,x - 1\frac{7}{8} + 2\,x = 7\frac{1}{2}$

28. $5\,x - 2\frac{3}{4} - x = 5\frac{1}{2}$

Equations Involving Negative Numbers

(1) Solve $n + 6 = -4$. You wish to have n alone on the left side. To get rid of $+6$, add -6 to both sides. What is $n + 6 - 6$? How much is $-4 - 6$?

SOLUTION. $\quad n + 6 = -4$

Adding -6, $\quad n = -10$

CHECK. $\quad -10 + 6 = -4$

(2) Solve $2\,n - 7 = -10$. First add $+7$ to both sides. What is $2\,n - 7 + 7$? What is $-10 + 7$? You then have $2\,n = -3$.

SOLUTION. $\quad 2\,n - 7 = -10$

Adding $+7$, $\quad 2\,n = -3$

Dividing by 2, $\quad n = -1\frac{1}{2}$

CHECK. $\quad 2(-1\frac{1}{2}) - 7 = -3 - 7 = -10$

(3) Look at all the following equations:

(*a*) $-n = 5$ (*c*) $-n + 1 = 3$ (*e*) $-2\,n - 3 = -5$

(*b*) $-n = -5$ (*d*) $-n - 5 = -2$ (*f*) $-2\,n + 7 = 3$

In all these equations the coefficient of n is negative. To make it positive and therefore easier to deal with, multiply both sides by -1. You will then have respectively:

(*a*) $n = -5$ (*c*) $n - 1 = -3$ (*e*) $2\,n + 3 = 5$

(*b*) $n = 5$ (*d*) $n + 5 = 2$ (*f*) $2\,n - 7 = -3$

Check the correctness of these statements.

Exercises

Solve and check the following equations: [1]

1. $-n = 6$

2. $-n = -7$

3. $-2\,n = -8$

4. $-3\,n = 24$

[1] If you have difficulty in checking these equations, do again the exercises on page 184.

5. $2n - 5n = 12$
6. $-3n - 4n = -21$
7. $5x = -30$
8. $7x = -35$
9. $-4x = 28$
10. $-8x = 72$
11. $-3x = -15$
12. $-5x = -20$
13. $y - 5 = 2$
14. $p - 8 = 6$
15. $n - 6 = -4$
16. $t - 4 = +4$
17. $t - 4 = -4$
18. $x - 8 = -12$
19. $p - 10 = +4$
20. $-6x = -18$
21. $-9x = -36$
22. $-2x = 4$
23. $-11x = 88$
24. $3x = -9$
25. $12x = -60$
26. $3x = -7$
27. $8x = -20$
28. $-4x = 9$
29. $-n + 3 = 7$
30. $-n + 3 = -7$
31. $-n - 3 = -7$
32. $-n - 3 = 7$
33. $x + 12 = 8$
34. $y + 9 = 4$
35. $p + 5 = -3$
36. $-6x = 15$
37. $-5x = -13$
38. $-3x = -17$
39. $2x = -3$
40. $5x = -4$
41. $-3x = -1$
42. $-2x = -1$
43. $-5x = 7$
44. $-3x = 5$
45. $t + 12 = -8$
46. $x - 8 = 4$
47. $y - 2 = +2$
48. $y + 2 = -2$
49. $8 - n = -2$
50. $-8 - n = -2$
51. $x + 5 = 5$
52. $\frac{2x}{3} = -7$
53. $\frac{3x}{2} = -5$
54. $-\frac{3}{2}x = 5$
55. $-\frac{2}{3}x = 3$
56. $-\frac{3}{5}x = -1$
57. $-\frac{5}{3}x = -2$
58. $2x + 5 = 3$
59. $3x + 7 = 1$
60. $3x - 1 = -10$
61. $4x - 5 = -21$
62. $4x - 1 = -7$

63. $6x - 5 = -20$
64. $2x + 9 = -2$
65. $3x + 7 = -4$
66. $3 - 2x = 8$
67. $5 - 3x = 11$
68. $5 - 2x = -1$
69. $3 - 5x = -7$
70. $-x + 5 = 2$
71. $-x - 5 = -3$
72. $-x + 7 = -5$
73. $-x - 11 = 1$
74. $-2x + 3 = -7$
75. $-2x - 5 = 9$
76. $-3x - 7 = 4$
77. $2(x - 3) = -10$
78. $2(x - 5) = -8$
79. $3(x + 5) = 9$
80. $4(x + 7) = 8$
81. $2(x - 5) = 3$
82. $4(x - 7) = 1$
83. $3(2x + 1) = 1$
84. $2(3x - 1) = 7$
85. $4(5x + 7) = 13$
86. $5x - 3(x + 2) = 4$
87. $4x - 3(x - 2) = -41$
88. $3x + 5(x + 2) = -6$
89. $7x - 3(3x + 1) = -1$
90. $5x + 2(1 - 4x) = 1$

Equations with the Unknown on Both Sides

(1) Try to solve the equation $5n - 2 = 2n + 13$.

Many equations in one unknown have the unknown on both sides of the equation and each side may have many terms — not just two as in the example above. All of them can be simplified, however, to the point where they do not have more than two terms on a side. The only new thing for you to learn, therefore, is to solve an equation like the one above.

(2) Solve $5n + 2 = 16 - 2n$. The new thing here is the $-2n$ on the right side of the equation. You do not wish it to be there; so add $+2n$ to both sides of the equation. (What is $5n + 2n + 2$? What is $16 - 2n + 2n$?) You then have $7n + 2 = 16$.

SOLUTION.

	$5n + 2 = 16 - 2n$
Adding $+2n$,	$7n + 2 = 16$
Adding -2,	$7n = 14$
Dividing by 7,	$n = 2$

CHECK. Left side: $5n + 2 = 5(2) + 2 = 10 + 2 = 12$

Right side: $16 - 2n = 16 - 2(2) = 16 - 4 = 12$

After a little practice you should be able to do steps such as adding $+2n$ and -2 in one step.

(3) Solve and check: $3(n-2)+5n=9n+12+n$.

First simplify each side as much as possible. You will get $8n-6=10n+12$. (Check the correctness of this.)

What number on the right side do you wish to get rid of? What number on the left side do you wish to get rid of? What do you do to get rid of $10n$? What do you do to get rid of -6?

SOLUTION.	$3(n-2)+5n=9n+12+n$
Removing parenthesis,	$3n-6+5n=9n+12+n$
Adding like terms,	$8n-6=10n+12$
Adding $-10n$,	$-2n-6=+12$
Multiplying by -1,	$2n+6=-12$
Adding -6,	$2n=-18$
Dividing by 2,	$n=-9$

CHECK.

Left side: $3(n-2)+5n=3(-9-2)+5(-9)=-33-45=-78$

Right side: $9n+12+n=9(-9)+12+(-9)=-81+12-9$
$=-78$

(Note that it would have been helpful to divide both sides by -2 instead of multiplying by -1.)

Exercises

Solve and check the following equations:

1. $2n+1=n+4$
2. $5n-1=3n+3$
3. $4n-5=2n+5$
4. $n+2=10-n$
5. $7n-2=4n+1$
6. $3n-6=14-n$
7. $n-6=18-3n$
8. $2n+4=n-2$
9. $3n-4=n-10$
10. $5n+7=2n-8$
11. $8n-5=7+9n$
12. $3n+2=7-2n$
13. $5n-3=9n-19$
14. $5n+2=3n+7$
15. $6n-1=2n+8$
16. $7n-3=4n+7$
17. $5n+6=3n+7$
18. $5n+7=3n+6$
19. $5n+7=3n+2$
20. $6n+8=2n-1$
21. $7x+2=4x+11$
22. $7x-2=4x+10$

23. $7x + 2 = 24 - 4x$
24. $7x - 2 = 20 - 4x$
25. $2 - 7x = 4x + 24$
26. $2 - 7x = 11 - 4x$
27. $-3x - 8 = 8x - 30$
28. $5y - 6 = 9y + 42$
29. $-6b + 11 = 2b + 43$
30. $x - 20 = 50 - 6x$
31. $14 = 2y + 20$
32. $-7a + 4 = 8a - 41$
33. $-2x - 7 = -8x - 19$
34. $5y - 3 = 8y - 16$
35. $5 + 2y = 0$
36. $0 = 4x + 20$
37. $8n - 12 = 2n$
38. $5n - 3 = 2n$
39. $3n + 9 = n$
40. $7n = 3 + 2n$
41. $x - 3 = 12$
42. $x - 3 = -12$
43. $-x - 3 = -12$
44. $-x - 3 = 12$
45. $2x + 5 = 13$
46. $2x + 5 = -13$
47. $-2x + 5 = 13$
48. $-2x + 5 = -13$
49. $2x - 5 = 13$
50. $2x - 5 = -13$
51. $-2x - 5 = 13$
52. $-2x - 5 = -13$
53. $2y - 5 + 8y = 11 + 7y$
54. $11x - 7 - 4x = 5x - 19$
55. $2x - 8 + 5x = 2x + 7$
56. $7b - 4 + b = 3b - 10$
57. $13 - 5x + 7 = 3 - 2x$
58. $9 + x - 6x = 5x + 21$
59. $3(n + 3) = 12$
60. $3(n - 3) = 12$
61. $4(n - 2) = 9$
62. $2(n + 1) = 5$
63. $5(n - 1) = 13$
64. $2(3n + 5) = 15$
65. $2(3n - 5) = -15$
66. $3(2n + 1) = 1$
67. $2(x + 10) = 42$
68. $5(y - 2) = 15$
69. $3(2b - 4) = 18$
70. $7x - (x - 4) = 25$
71. $-5y - (2 - y) = 18$
72. $-(6 - 2x) = 24$
73. $16 = (2x + 4)$
74. $(2x + 3) = -17$
75. $4x + 5(x + 2) = 46$
76. $3(n - 1) = 2(n + 2)$
77. $2(2n + 1) = 3(n - 5)$
78. $7n - 5 = 2(2n + 6)$
79. $5n + 3 = 2(n - 1)$
80. $4(3n + 1) = 3(2n + 1)$
81. $3(n + 1) - 15 = 2n - 10$
82. $6(n - 4) - 15 = 3(n - 8)$

83. $2(n+1)-3n=3(2n+3)$

84. $2(x-3)+3(x-2)=8$

85. $5b+2(4-b)=32$

86. $-3x+6(x-4)=9$

87. $-7b+4(2b-3)=16$

88. $5b-3(4-2b)=2b+42$

89. $6(x-3)-4(x+2)=4-x$

90. $7(b-2)-2(3+b)=0$

91. $(x+3)(x-3)=x^2+3x-6$

92. $(2x+3)^2=(x+5)(4x-3)-1$

Different Forms of the Same Formula

It is often convenient to use a formula in a different form from that in which you learned it. For example, if you knew the length and width of a rectangle and wished to find the area, you would use $A = lw$. But if you knew the area and length and wished to find the width, it would be more convenient to use another form of the same formula.

Which of the formulas below would you use if you knew the area and length of a rectangle and wanted to find the width?

If you knew the area and the width and wanted to find the length, which formula would you use?

(*a*) $A = lw$ (*b*) $l = \frac{A}{w}$ (*c*) $w = \frac{A}{l}$

The three formulas above really express only one relation between the *area, length,* and *width* of a rectangle. They are different forms of the same equation and you could probably decide quite accurately what changes you could make in it because it is simple and you are familiar with the relations expressed by it.

However, some of the formulas which you will later meet in science, mathematics, industrial arts, and general reading are too difficult to be changed about by a "common-sense judgment." Fortunately mathematics furnishes you precise tools that make it possible for you to write all the forms of a formula provided that you know one form only. These tools are the axioms which you have already studied in connection with equations. You will now see how useful the axioms are in changing the forms of formulas.

Changing Formulas into Other Forms

(1) We say that the formula $A = lw$ is *solved* for A; the formula $l = \frac{A}{w}$ is solved for l; the formula $w = \frac{A}{l}$ is solved for w. Read each of the formulas below and state for which letter each is solved:

(*a*) $p = a + b + c$ (*b*) $a = p - (b + c)$ (*c*) $c = p - (a + b)$

(2) Solve the formula $A = lw$ for l. Think as follows: What has been done to the l? (It has been multiplied by w.) How shall I get l alone? (Divide lw by w. Why do I divide, instead of subtract?) If I divide one side of the equation by w I must divide the other side of the equation by w. How do I write $A \div w$?

Arrange the work vertically.

SOLUTION. $A = lw$

Dividing by w, $\frac{A}{w} = \frac{lw}{w}$

$\frac{A}{w} = l$ or $l = \frac{A}{w}$

(The second step may be done mentally and not written.)

(3) Solve the formula $S = M + C$ for M. What has been done to the M? How shall I get M alone? If I subtract C (or add $-C$) from one side of the equation, what must I do to the other side of the equation?

SOLUTION. $S = M + C$

Adding $-C$, $S - C = M + C - C$

$S - C = M$, or $M = S - C$

(4) Solve the equation $p = 2l + 2w$ for l. First, you wish to have all terms containing l (in this case only one term) on one side of the equation and all other terms on the other side. To get rid of $2w$, add $-2w$ to both sides.

SOLUTION. $p = 2l + 2w$

Adding $-2w$, $p - 2w = 2l$

Dividing by 2, $\frac{p - 2w}{2} = l$ or $l = \frac{p - 2w}{2}$

(5) State what has been done to each side of the equation in changing the form of each of the following:

(*a*) If $c = \pi d$, then $d = \frac{c}{\pi}$.

(*b*) If $A = \frac{bh}{2}$, then $bh = 2A$.

(*c*) If $x + y = 12$, then $x = 12 - y$.

(*d*) If $p = 2b - 2h$, then $2b = p + 2h$.

(*e*) If $a - b = c$, then $c + b = a$. (Note that this statement indicates a check for subtraction.)

(*f*) If $\frac{a}{b} = c$, then $bc = a$. (This indicates a check for _?_.)

Exercises

1. Divide each side of the formula $lw = A$ by l. What new formula do you obtain?

2. Divide each side of the formula $V = lwh$ by lw. What new formula do you obtain?

3. Multiply each side of the formula $A = \frac{bh}{2}$ by 2. What new formula do you obtain? Solve it for b.

4. Subtract b from both sides of the formula $p = 2a + b$. Then solve for a.

5. Divide each side of the formula $c = \pi d$ by π.

6. What has been done to the formula $A = bh$ to obtain the formula $h = \frac{A}{b}$?

7. What must be done to both sides of the formula $d = 2r$ to obtain the formula $r = \frac{1}{2}d$?

8. What axiom has been used to change the formula $S = a + b$ into the formula $a = S - b$?

Solve the following for x:

9. $rx = d$

10. $x + m = n$

11. $x - m = n$

12. $\frac{x}{m} = n$

13. $2\pi rx = S$

14. $l^2x = V$

Using Formulas

When you use a formula by substituting numbers for all but one letter, there are two ways to find the value of the desired letter. (1) You can substitute the numbers in the formula first and solve the resulting equation, or (2) you can solve the formula for the desired letter first and then substitute the numbers.

Exercises

Using the formula given, evaluate the letter as indicated:

1. $A = lw$; $A = 36$ sq. ft., $l = 12$ ft., $w =$ _?_
2. $A = lw$; $A = 7.5$ sq. in., $w = 2.5$ in., $l =$ _?_
3. $p = a + b + c$; $p = 19$ ft., $a = 7$ ft., $c = 6$ ft., $b =$ _?_
4. $p = a + b + c$; $p = 21\frac{1}{3}$ yd., $b = 9\frac{1}{2}$ yd., $c = 6\frac{1}{2}$ yd., $a =$ _?_
5. $p = 2a + b$; $p = 20$ cm., $b = 4$ cm., $a =$ _?_
6. $p = 2a + b$; $p = 18\frac{1}{2}$ yd., $b = 4\frac{1}{3}$ yd., $a =$ _?_
7. $p = 2a + b$; $p = 26$ in., $a = 8$ in., $b =$ _?_
8. $p = 2a + b$; $p = 2.34$ ft., $a = .84$ ft., $b =$ _?_
9. $p = rb$; $p = 24$, $b = 48$, $r =$ _?_ %
10. $p = rb$; $p = 28$, $b = 84$, $r =$ _?_ %
11. $p = rb$; $p = 75$, $r = .05$, $b =$ _?_
12. $p = rb$; $p = 160$, $r = .03$, $b =$ _?_
13. $V = lwh$; $V = 42.3$ cu. in., $l = 7.2$ in., $h = 2.4$ in., $w =$ _?_ (to the nearest tenth)
14. $V = l^2h$; $V = 96$ cu. ft., $l = 5.6$ ft., $h =$ _?_ (to the nearest tenth)
15. $V = \pi r^2h$; $V = 125$ cu. in., $r = 6.2$ in., $h =$ _?_ (to the nearest tenth)
16. $A = \frac{1}{2}h(a + b)$; $A = 48$, $a = 3$, $b = 4$, $h =$ _?_
17. $A = \frac{1}{2}h(a + b)$; $A = 60$, $h = 8$, $a = 3$, $b =$ _?_
18. $C = np$; $C = 40$, $n = 12$, $p =$ _?_
19. $V = l^2h$; $V = 96$, $l = 4$, $h =$ _?_ .

Literal Equations

In the early work of this chapter you were dealing with *numerical equations* of the first degree in one unknown; that is, ones in which only the unknown number is represented by a letter. For example, $3x = 9$ and $2x - 5 = 10$ are numerical equations where only the unknown number is represented by a letter — in this case, x.

Literal equations are ones which have at least one number in addition to the unknown represented by a letter. Thus, $3x = b$, $2x - c = 10$, and $2x - c = k$ are examples of literal equations. Literal equations are solved in the same way as formulas.

Exercises

Solve each equation for n. You need not check.

1. $n + a = b$
2. $an = b$
3. $n - a = b$
4. $\frac{n}{a} = b$
5. $a = bnc$
6. $2n + a = b$
7. $2an = b$
8. $2n - a = b$
9. $2n + 5a = 8a$
10. $n - p = 3q$
11. $3a + n = 5b$
12. $6a + 2b = 2n$
13. $n - 5a = 4$
14. $2n - b = 5a$
15. $n + a = b + c$
16. $an = b + c$
17. $\frac{n}{a} = b + c$
18. $n - a = b + c$
19. $2a + 2n = b + c$
20. $-n = b$
21. $-n = -4b$
22. $-n - a = -b$
23. $-n + a = b$
24. $an + ac = b$
25. $\frac{2n}{a} = b$
26. $2(n + a) = b$
27. $a(n - b) = c$
28. $a(b - n) = c$
29. $5(n + 2) = 11$
30. $3(n + 6) - 12 = 14$
31. $3(3n - 1) = 4(2n + 1)$
32. $3(2n + 1) = 4n$
33. $\frac{n + 2}{a} = a - 3$

Historical Note

Algebra, as we know it, had its beginnings in the seventeenth century. In crude form, however, it was known to the ancient Egyptians. It was greatly improved by the Arabs through whom it was introduced into Europe largely through the Moorish universities in Spain. Like arithmetic (and unlike geometry), it is based on counting and numbers, and the literal numbers are a great advance over the particular numbers of arithmetic.

The first known record of an equation solved is contained in an old papyrus manuscript of about 1700 B.C. which was copied from one written about 200 years earlier. The title of this papyrus is *Rules for Enquiring into Nature, and for Knowing All That Exists, Every Mystery, Every Secret.* This is one of the oldest books on mathematics left to us. In it, what little symbolism is used is crude and the method of solution of equations is clumsy. Most of the solutions are written out fully in words. You can solve equations now much more simply than could the greatest mathematicians of ancient times.

Little was done to improve the notation of algebra or the method of solution of equations until the time of Diophantus of Alexandria, who lived about 275 A.D. He introduced a symbolism that was far superior to any used before, and he was able to write equations much as we do. Although his methods of solving equations were decidedly in advance of those used earlier, they were still awkward as compared to present methods. The definite recognition of the use of the axioms in simplifying equations is attributed to Arab writers — among them Al-Khwârizmî, who lived about 825 A.D.

Chapter Summary

A *first-degree equation in one unknown* is one in which the unknown number occurs only with the exponent 1 and does not occur in the denominator of any fraction. Such an equation may be either an identical equation or a conditional equation. An *identical equation* is one that is true for all values of the unknown number. A *conditional equation* is one that is true for only particular values of the unknown number. Solving an equation is finding the value of the unknown which *satisfies* the equation; that is, the value or *root* which makes the two sides of the equation equal.

In this chapter you first analyzed the methods you had previously used in solving equations with the unknown on one side only. You saw that to solve an equation *you must operate on both sides of the equation.* If you add a number to one side of an equation, you have to add the same number to the other side in order to keep the balance. This is true for the other operations: subtraction, multiplication, and division. Knowledge of inverse processes is valuable in indicating which process to use in solving an equation.

This analysis helped you to learn how to solve equations with the unknown on both sides, by using the *axioms of the equation.* By means of the *addition and subtraction axioms* you can get all the terms containing the unknown on one side of the equation and the arithmetic numbers on the other side. Then by means of the *multiplication and division axioms* you can complete the solution. Ability to operate with signed numbers is indispensable in solving many equations.

Formulas may have several forms, all of which state the same relationship between the variables. It is necessary to remember only one of the forms. The other forms may be obtained from the one you remember by use of the equation axioms. In solving for one unknown in a formula when the others are given, you have a choice of substituting the values first and then solving the resulting equation or solving for the unknown letter first and then substituting the given values.

Literal equations are solved by the same methods as numerical equations.

You should know the meaning of these technical terms:

Equation of the first degree in one unknown

Identical equation	Equation axioms
Conditional equation	Literal equation

Chapter Review

1. State four equation axioms.

2. Is $3(a + 1) = 3a + 3$ an identical or a conditional equation?

3. Is $3(a + 1) = 2a + 2$ an identical or a conditional equation?

4. Is $3(a + 1) = 3a + 2$ a true statement for any value of a?

5. When is an equation satisfied?

6. Is 5 a root of $7n + 3 = 4n - 2$? Explain.

7. State what you do to each side of the equation in proceeding from step to step in the solution of $2n + 7 = 5n - 10$.

8. What is the root of $2n + 3 = 9$? Add any number you please to both sides of the original equation. In this new equation, substitute the root you found. What do you discover? Multiply both sides of the original equation by any number you please. Again substitute the root. What do you discover?

9. What must you do to both sides of the equation $4x^2 + 12x + 8 = 2(x + 1)$ to obtain $2x^2 + 6x + 4 = x + 1$?

10. What is a convenient first step in solving $-2n - 5 = 3$ in order to make the coefficient of n positive?

11. What two steps do you take in solving $\frac{2n}{3} = 7$?

12. What should you do first in solving $6n + 3 + 2n - 5 = 7 + 3n - 4$?

13. What should you do first in solving $2(n + 3) - 4 = 5 - 7(2n - 1)$?

14. In checking a problem, should you substitute your answer in the equation you have used to get the answer or should you see if the answer checks in the original problem? Explain.

15. Which of the following equations are equations of the first degree in one unknown?

(*a*) $2n - 3 = n + 8$
(*b*) $2x + 3y = 42$
(*c*) $x^2 - 3x + 2 = 0$
(*d*) $\frac{2}{n} + n = 5$
(*e*) $2(n - 3) + 4 = 15$

Solve and check the following equations:

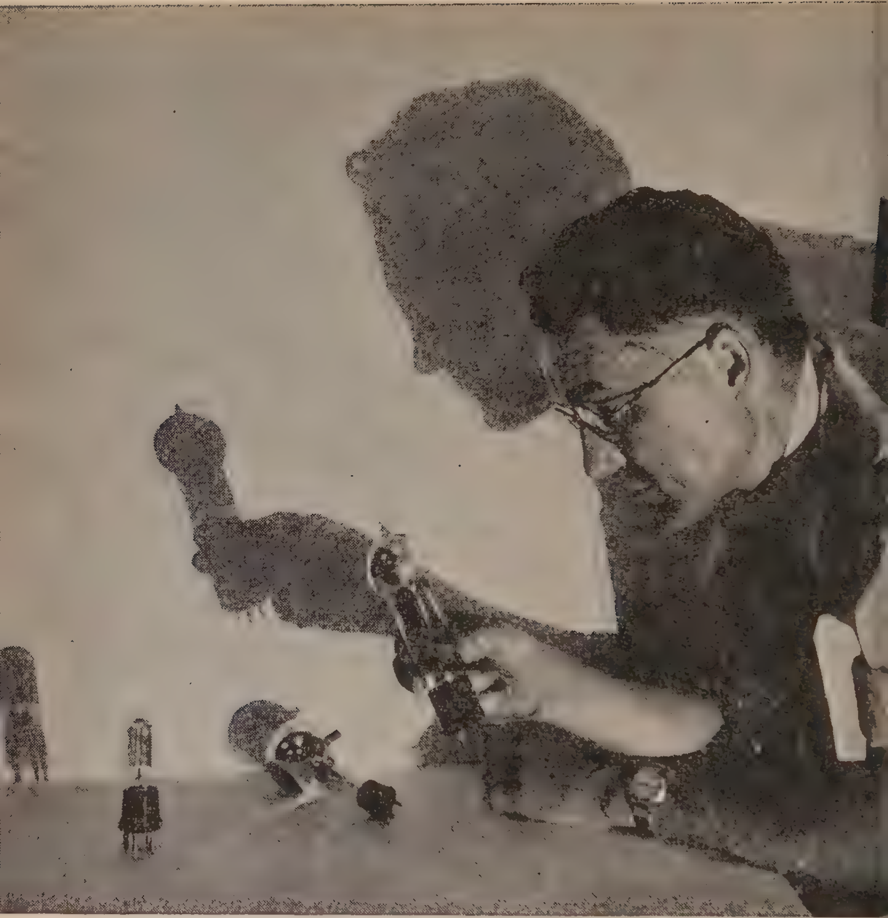
16. $n - 7 = -2$
17. $-n + 7 = -2$
18. $n + 8 = -5$
19. $2n = 3$
20. $3n = 2$
21. $-2n = 3$
22. $-2n = -3$
23. $2n + 5 = 4$
24. $2n - 5 = 4$
25. $\frac{3x}{5} = 4$
26. $3n + 5 = n - 7$
27. $5n - 1 = 2n + 5$
28. $4n + 5 + 2n = 7 + n - 5$
29. $2(2n - 3) - 5(2n + 3) = 10n - 5$
30. $5(n + 1) - 3n = 3(2n + 3)$

Solve the following equations for n:

31. $n - b = a$
32. $bn = a$
33. $\frac{n}{a} = b$
34. $\frac{a}{n} = b$
35. $3n + b = a$
36. $2bn = a$
37. $3n - b = a$
38. $3(n + b) = a$
39. $\frac{2n}{b} = a$
40. $\frac{2b}{n} = a$

41. If $p = 2l + 2w$, $p = 42$, and $l = 10$, what is the value of w?

42. If $V = \pi r^2 h$, $V = 628$, and $r = 5$, what is the value of h? (Use $\pi = \frac{22}{7}$. Record answer to nearest tenth.)



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This engineer finds his problems in the vacuum tubes with which he works. He must be able to discover relationships and formulate equations to express them. Your study of the analysis and solution of verbal problems is direct training for attacking problems in this same way.

Chapter X

VERBAL PROBLEMS AND THEIR SOLUTION

Verbal problems are problems that are stated in words. They are introduced into the course in algebra because the solution of them gives practice in discovering relationships between quantities and in expressing the quantities and their relationships algebraically. In the solution of verbal problems we translate the statements that are made in words into the language of algebra. Then we express the statements in the form of equations and by solving the equations we arrive at the conclusions that can be drawn from them. In other words, we solve the problem by means of algebra. This is what the persons do who use mathematics in commerce, industry, engineering, architecture, and so on.

With the general procedure used in solving verbal problems you are already familiar. In this chapter a more careful analysis of the method used will be made and practice in the use of the method will be given so that you can attack problems with increased confidence. The three skills needed in problem solution are the ability to —

1. Express one quantity in terms of another,

2. Formulate equations that express relations between quantities,

3. Solve equations.

Before going into the methods of problem analysis, you need further practice in the first two of these skills to make sure you have mastered them.

Expressing One Quantity in Terms of Another

1. Let n represent a certain number, and then express the following quantities in terms of n:

(a) 7 times the number
(b) 7 more than the number
(c) 7 less than the number
(d) the number divided by 7
(e) 7 more than twice the number
(f) 7 less than twice the number
(g) 7 more than one half the number

2. Let $2n$ represent a certain number, and then express the quantities in Ex. 1 in terms of n.

3. Let $n + 6$ represent a certain number, and then express the quantities in Ex. 1 in terms of n.

4. If n represents a man's age now, what represents his age 10 years from now? 10 years ago? m years from now?

5. If $3n$ represents a man's age, what represents his age 5 years from now?

6. If $n + 3$ represents a boy's age, what represents the age of the father, who is twice as old?

7. If $n - 3$ represents a girl's age, what represents the age of her mother, who is 1 year more than twice as old?

8. If clover seed is 20 cents a pound, what represents the cost in cents of n pounds?

9. What represents the cost in cents of $(n - 2)$ postage stamps at 3 cents a stamp?

10. What is the value in cents of n dimes? of $3n$ dimes? n quarters? $2n$ nickels? $4n$ half dollars? $(2n + 3)$ quarters?

11. A train runs for t hours. Express the distance it will cover at the rate of —

(a) 35 miles an hour
(b) m miles an hour
(c) $(r + 6)$ miles an hour
(d) $5r$ miles an hour

12. A soda clerk sold 200 drinks, some at 5 cents and some at 10 cents each. Represent the number of drinks at 5 cents by n. Express the following in terms of n:

(*a*) The number of drinks at 10 cents

(*b*) The number of cents received for all the 5-cent drinks

(*c*) The number of cents received for all the 10-cent drinks

(*d*) The total number of cents received

13. A stenographer worked 100 hours, part of the time at 40 cents an hour and part at 60 cents. Represent the number of hours at 40 cents by n. Express the following in terms of n:

(*a*) The number of hours at 60 cents

(*b*) The number of cents received for all the 40-cent hours

(*c*) The number of cents received for all the 60-cent hours

(*d*) The total number of cents received

14. A father is 9 times as old as his son. Would you represent the father's age or the son's age by n? (Choose the one which will make the work easier; in this case, the son's age.) Represent the following in terms of n:

(*a*) The father's age

(*b*) The son's age 8 years from now

(*c*) The father's age 8 years from now

15. A, B, and C receive a certain amount of money. B receives \$100 less than A and C \$200 more than A. Which of these amounts would you represent by n? (Choose the one which will make the work easiest; in this case, A's amount, because as the statement is written it is least dependent upon the others.) Express in terms of n the amounts received by the others.

16. A man gave a certain amount of money to each of three sons. The first received twice as much as the third and the second received \$100 more than the third. Represent one of the amounts by n and express the other amounts in terms of n.

Expressing Relationships as Equations

Verbal problems are solved by means of equations, but the equations are not given to you. You have to make your own equations from the relationships given in the problem. In making equations from statements that are given, be sure to analyze the statements carefully. For example, to write the equation for the statement, "a is 3 more than b," think: Which is the larger, a or b? In this case it is a. With this answer clearly in mind you may express the relationship in one of three ways:

1. If you subtract 3 from a, the result will equal b.

$$a - 3 = b.$$

2. If you add 3 to b, the result will equal a.

$$a = b + 3.$$

3. If you subtract b from a, the result is 3.

$$a - b = 3.$$

Exercises

Write the following statements of Exs. 1 to 28 as equations:

1. t is 10 more than s.

2. $2n$ is 3 more than n.

3. $2n - 3$ is 6 more than n.

4. a is 5 less than b.

5. n is 6 less than $3n$.

6. $2n - 3$ is 6 less than $4n$.

7. b is 5 less than a.

8. $3n$ is 6 less than n.

9. $4n$ is 6 less than $2n - 3$.

10. The difference between a and b is 3 (a being greater).

11. The difference between $2n$ and n is 3 ($2n$ being the greater).

12. The difference between $2n - 3$ and n is 6 ($2n - 3$ being the greater).

13. The quantity $4n + 4$ is equal to the quantity $3(n + 4)$.

14. The result of subtracting 7 from $3n$ is the same as adding 5 to n.

15. a is 5 times b.

16. $n + 15$ is 4 times n.

17. The quantity $2n + 3$ is 3 times the quantity $2n - 3$.

18. The quantity $2n + 5$ is 6 less than the quantity $4n - 3$.

19. a is 4 more than 3 times b.

20. The quantity $7n - 4$ is 4 more than 3 times n.

21. Twice the quantity $n - 3$ is 5 more than n.

22. Add 11 to $2n + 2$, then subtract 8 from n, and state that the results are equal.

23. $3n - 4$ exceeds $n + 2$ by 6.

24. The sum of a, b, and c is 150.

25. The sum of n, $n - 100$, and $2n - 600$ is 4000.

26. a exceeds twice b by 12.

27. Twice n added to twice the quantity $n + 2$ is 36.

28. 15% of the quantity $n + 2$ is 42 more than 12% of n.

29. The length of a rectangle is 5 inches more than the width. Express the length and the width in terms of n. Write an equation stating that the perimeter is 48 inches.

30. A is 6 times as old as B. Express both ages in terms of n. Express the age of each in 20 years. Write an equation stating that in 20 years A will be twice as old as B.

31. Express the value in cents of —

(*a*) n nickels

(*b*) $(3 - n)$ dimes

(*c*) $(n + 5)$ quarters

(*d*) $(12 - n)$ half dollars

State algebraically that the total value is $6.95.

32. A slow plane travels at the rate of r miles an hour; another plane travels 80 miles an hour faster. Express algebraically —

(*a*) the rate of the fast plane

(*b*) the distance passed over by each in 5 hours

(*c*) the fact that the sum of the distances covered by the two planes in this time is 1900 miles

33. The sum of two numbers is 20. What is the first number if the second is n? Write an equation stating that the second number is 4 more than the first.

34. There are 36 pupils in a mathematics class. How many boys are there if there are n girls? Write an equation stating that the number of boys is 2 less than the number of girls.

35. What is the cost in dollars of n pounds of butter at 38 cents a pound? of $(48 - n)$ pounds at 50 cents? (State that the total cost of both kinds is \$24.00.)

36. A farmer has 100 pounds of a mixture of clover seed and bluegrass seed.

(*a*) Express algebraically the number of pounds of clover seed if there are n pounds of bluegrass seed; the value of the bluegrass seed (n pounds) at 15 cents a pound; and the value of the clover seed at 20 cents a pound;

(*b*) State by an equation that the value of both kinds together is \$19.

37. A mother's age is 20 years less than 5 times her daughter's age. Express (*a*) their present ages in terms of n and (*b*) the age of each 2 years ago. State by an equation that 2 years ago the mother's age was 4 times that of the daughter.

Analysis and Solution of a Problem

The solution of any verbal problem solved by means of one unknown consists in (1) analyzing it to recognize the relationships between the numbers in it, (2) using these relationships to express all the unknown quantities of the problem in terms of one unknown and to write an equation, and (3) solving the equation for the values of the unknown numbers. The method of carrying out these processes is illustrated below.

EXAMPLE. A grocer has two kinds of tea, one kind selling at 80 cents a pound and the other at 60 cents a pound. How many pounds of each kind must he use to make 50 pounds that he can sell at 72 cents a pound?

First, make sure that you understand what the problem is. Go through the motions of what the grocer has to do. He has two containers full of tea and an empty container. He takes some tea from the 80-cent container and puts it into the empty container. He does the same thing with tea from the 60-cent container. But he has to take just the right amount of each to make 50 pounds worth 72 cents a pound.

Would it be right to take 1 pound of the 80-cent tea and 49 pounds of the 60-cent tea? 1 pound of the 60-cent tea and 49 pounds of the 80-cent tea? Give reasons for your answers.

First make a guess at your answer and check to see if you are right. Suppose it would take 40 pounds of 80-cent tea.

There would then be 10 pounds of 60-cent tea. Why?

The 40 pounds of 80-cent tea is worth 3200 cents.

The 10 pounds of 60-cent tea is worth 600 cents.

The total value of the two kinds together is 3800 cents. Since the value of the mixture should be 3600 cents, our guess is incorrect. But we see just what to do to solve the problem. Instead of guessing 40 pounds of 80-cent tea, we can write n pounds of 80-cent tea and work through the same relationships.

As you gain experience in problem solving, you should be able to make the analysis without the first step of guessing or trying a number. Whenever you are in doubt about the method, this first step will help you.

The quantities we work with are listed below.

Number of pounds of 80-cent tea	n
Number of pounds of 60-cent tea	$50 - n$
Total value of 80-cent tea	$80\,n$
Total value of 60-cent tea	$60(50 - n)$
Intended value of the mixture	3600

Copy this list. Let the number of pounds of 80-cent tea be n and fill in the spaces at the right as is done above. (If you have difficulty in doing this, refer to your method of checking your 40-pound guess.) If n is to be the right number to fit the conditions of the problem, $80\,n + 60(50 - n)$ must equal 3600. (This is the final relationship you saw in checking your guess.) Make the equation and solve.

SOLUTION.

$$80\,n + 60(50 - n) = 3600$$
$$80\,n + 3000 - 60\,n = 3600$$
$$20\,n = 600$$
$$n = 30 \qquad 50 - n = 20$$

ANSWER. 30 pounds of 80-cent tea, 20 pounds of 60-cent tea.

CHECK. 30 pounds of 80-cent tea plus 20 pounds of 60-cent tea should have the same value as 50 pounds of the mixture at 72 cents a pound. $30(80) + 20(60) = 2400 + 1200 = 3600$

Below is a summary of the steps in problem solving as you have been using them. They represent a general method of solution applicable to many types of problems. Their value is that if you follow the plan outlined you do not have to learn a new method for every particular type of problem which you encounter.

Step 1. Read the problem until you are sure you know what it means. If it is a type of problem new to you, it is helpful in this connection to assume an answer and check to see if you are right. In the checking you will discover while working with arithmetic numbers just what the quantities are that you must work with and the relationships between them.

Step 2. Make a list of the quantities with which you will work.

Step 3. Find as many relationships between these quantities as there are quantities. For example, if there are three quantities, there must be three relationships in order to get a solution.

Step 4. Select the quantity in terms of which you can most easily express the other quantities. Represent this quantity by n or some other suitable algebraic expression.

Step 5. Express the other quantities in terms of this quantity by means of the relationships in Step 3. In doing this you will use all but one of the relationships.

Step 6. Using the remaining relationship, write an equation.

Step 7. Solve the equation.

Step 8. Check by making sure that your answer fits all the conditions of the problem.

Exercises

1. Of three numbers, the first is 3 times the second and the third is 5 more than the second. The sum of the first and third is 41. Find the three numbers.

2. Of three numbers, the first is 5 less than 3 times the third and the second is 10 more than the third. The first is 2 less than the second. Find the numbers.

3. A is 2 more than B. If A is multiplied by 25 and B is multiplied by 50, the sum of the resulting numbers is 500. Find A and B.

4. The first of two numbers is 3 times the second. If 15 is added to each, the first result is 2 times the second result. What are the two numbers?

5. A father is 9 times as old as his son. In 9 years he will be only 3 times as old. What is the age of each now?

6. The length of a tennis court for singles is 3 feet shorter than 3 times its width. Find the length and the width of the court if the perimeter is 210 feet.

7. A boy paid 48 cents for 20 stamps; some cost 2 cents each, and the rest cost 3 cents each. How many of each did he buy?

8. A's present age is twice B's age; 10 years ago A's age was 3 times B's age. Find the age of each now.

9. A's present age exceeds B's age by 25 years. In 15 years he will be twice as old as B. Find their present ages.

10. A purse was found that contained 20 coins, all nickels and dimes. Find the number of each if the combined value was $1.60.

11. The length of a rectangle is 2 inches more than the width. The perimeter is 40 inches. Find the length and the width.

12. The perimeter of a triangle, ABC, is 52 inches. AB is 6 inches less than BC and BC is 8 inches more than AC. Find the length of each side.

13. Divide \$920 among A, B, and C so that A receives 5 times as much as C and 3 times as much as B.

14. A man is now 45 years old and his son is 15. In how many years will he be twice as old as his son?

15. I received at a candy counter twice as many dimes as quarters and 6 more nickels than the dimes and quarters together. How many of each coin did I receive if the value of all was \$7.50?

16. Eighteen coins, all dimes and quarters, amount to \$2.25. Find the number of each kind of coin.

17. A is now 17 years old and B is 50. In how many years will A be one half as old as B?

18. The value of 36 coins, all dimes and quarters, is \$6.60. Find the number of coins of each kind.

19. A collection of nickels, dimes, and quarters amounts to \$4. There are 10 more nickels than dimes and 2 less quarters than dimes. Find the number of each.

20. A man is now 40 years old and his son is 14 years old. In how many years will the father be twice as old as his son?

21. One hundred pounds of bacon, part at 27 cents and part at 35 cents, cost \$31.48. How many pounds of each kind are there?

22. A father's age is 32 years and his daughter's age is 2 years. He is now 16 times as old as his daughter. In how many years will he be only 4 times as old?

23. A newspaper boy paid a bill of \$6.80, using nickels, dimes, and quarters. He used the same number of each kind of coin. How many coins did he use in all?

24. How many pounds of tea at 60 cents a pound must be added to 30 pounds of tea at 75 cents a pound to make a mixture worth 70 cents a pound?

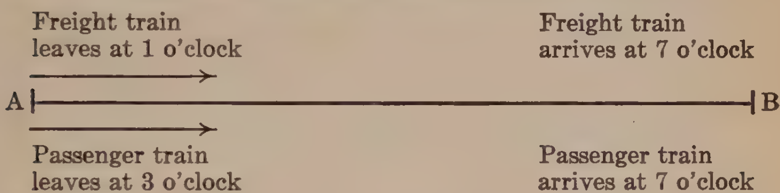
25. A storekeeper wishes to mix two kinds of candy which he ordinarily sells at 30 cents and 45 cents a pound so that he can sell the mixture for 40 cents a pound. How much of each kind should he use to make 60 pounds of the mixture?

Time-Rate-Distance Problems

The following exercises will help you in analyzing problems concerning time, rate, and distance. The method of solution of such problems is, of course, the same as the one you have been using on the problems of other types.

In solving time-rate-distance problems, drawing lines as is done below will often aid you in the analysis. Read (1), (2), and (3) below and then decide to which of them each statement at the bottom of this page from (*a*) to (*f*) applies.

(1) A freight train leaves point *A* at 1 o'clock; a passenger train leaves the same point at 3 o'clock, going in the same direction. The passenger train overtakes the freight train at 7 o'clock (that is, they both reach a point *B* at this time).



(2) A freight train and a passenger train leave point *A* at 1 o'clock, going in opposite directions. At 7 o'clock they are 450 miles apart.

(3) A passenger train leaves point *A* at a certain time. Three hours later a freight train leaves the same point, going in the same direction. When the passenger train reaches a point *B*, the freight train is 450 miles behind.

(*a*) The difference between the distances is 450 miles.

(*b*) The distances are equal.

(*c*) The sum of the distances is 450 miles.

(*d*) The time traveled by each is the same.

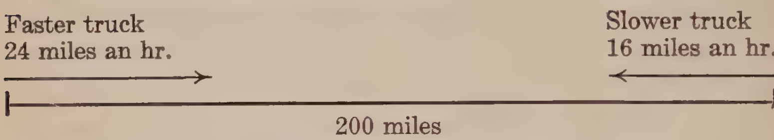
(*e*) The time traveled by the freight train is 3 hours less than the time traveled by the passenger train.

(*f*) The time traveled by the passenger train is 2 hours less than the time traveled by the freight train.

(4) Now study the example on the next page and explain each step.

EXAMPLE. Two auto trucks 200 miles apart travel toward each other at rates of 24 and 16 miles an hour. The slower truck has a breakdown soon after starting and is delayed an hour. In how many hours will they meet?

Read the problem carefully. Draw a line to help you *see* the distances and direction of travel.



You have the following relationships in the problem:

(*a*) The distance of the slower truck equals its rate times its time.

(*b*) The distance of the faster truck equals its rate times its time.

(*c*) The time the slower truck is traveling is one hour less than the time the faster truck is traveling.

(*d*) The sum of the distances is 200 miles.

SOLUTION.

Time of faster truck	n
Time of slower truck	$n - 1$
Distance of faster truck	$24\,n$
Distance of slower truck	$16(n - 1)$

$$\begin{aligned} 24\,n + 16(n - 1) &= 200 \\ 24\,n + 16\,n - 16 &= 200 \\ 40\,n &= 216 \\ n &= 5\tfrac{16}{40} \text{ or } 5\tfrac{2}{5} \\ n - 1 &= 4\tfrac{2}{5} \end{aligned}$$

ANSWER. 5 hours, 24 minutes

CHECK.

Faster truck travels $5\frac{2}{5}$ hours at 24 miles an hour $= 129\frac{3}{5}$ miles.
Slower truck travels $4\frac{2}{5}$ hours at 16 miles an hour $= 70\frac{2}{5}$ miles.
Both trucks travel $129\frac{3}{5} + 70\frac{2}{5}$ miles $= 200$ miles.

Note how picturing the quantities we have on a straight line, as was done above, helps to make clear the relationships that should be stated in the equation. From the diagram you should see at once that the distance both trucks travel is $24\,n + 16(n - 1) = 200$ miles. This is the equation needed to solve the problem.

Exercises

1. Two trains leave Chicago at the same time — one eastbound, the other westbound. The eastbound train travels 10 miles less in an hour than does the westbound. Express —

(*a*) the rate of each

(*b*) the distance traveled by each in 4 hours

(*c*) the fact that the trains were 440 miles apart at the end of four hours

2. A freight train traveling at the rate of 30 miles an hour is followed 2 hours later by an express train traveling at the rate of 50 miles an hour. In how many hours will the express train overtake the freight train?

3. Two automobiles start at the same time at the same place and travel in opposite directions. One travels at the rate of 45 miles an hour and the other at the rate of 40 miles an hour. In how many hours will they be 255 miles apart?

4. In Ex. 3, if the two automobiles travel in the same direction, in how many hours will they be 40 miles apart?

5. Two boys start from the same place at the same time, riding bicycles. One rides at the rate of 8 miles an hour and the other at 5 miles an hour. They go in the same direction. In how many hours will they be 15 miles apart?

6. If, in Ex. 5, the faster boy starts 1 hour later than the other, in how many hours will he overtake the slower boy?

7. One car running 40 miles an hour left a certain place 4 hours later than another car running in the same direction at the rate of 30 miles an hour. In how many hours will the faster car overtake the other?

8. A and B start from the same place and travel in opposite directions. A's rate of travel is twice B's. In 4 hours they are 150 miles apart. Find the rate of travel of each.

9. A plane leaves a certain port at 10 o'clock, flying due north. Another plane starts from the same port at 12 o'clock, flying due south. At 2 o'clock they are 1200 miles apart. Determine how fast each travels, if the rate of the first is one half that of the second.

10. A left a town 4 hours after B left. They traveled in opposite directions. A traveled at the rate of 12 miles an hour and B traveled at the rate of 20 miles an hour. In how many hours will they be 272 miles apart?

11. A bicyclist had been traveling 15 miles an hour for 8 hours when he was overtaken by an automobile that had left the same starting point 5 hours after he left. Find the rate of speed of the automobile.

12. Two autoists 240 miles apart start toward each other at the same time. The faster one travels 50 miles an hour and the slower one travels 35 miles an hour. How long will it be before they meet if the faster is delayed 2 hours on the trip?

13. A starts from a certain place, traveling at the rate of 4 miles an hour. Five hours later B starts from the same place and travels in the same direction at the rate of 6 miles an hour. In how many hours will B overtake A?

14. An airplane is 75 miles directly behind a ship sailing a straight course. The rate of the plane is 120 miles an hour and that of the ship is 20 miles an hour. How long will it take the plane to overtake the ship?

Problems Involving Per Cents

Many problems involving per cents are made easier by the use of equations. The analysis is in general the same as for other problems.

(1) 24 is what per cent of 62? Assuming an answer and checking to see if you are right will show you what to do. Suppose the answer is 40%. To check, you will find 40% of 62 to see if you get 24. Does .40(62) equal 24? To solve the problem, then, simply say that $n\%$ of 62 must be 24. ($n\%$ is written $.01\,n$.)

SOLUTION.
$$.01\,n(62) = 24$$
$$.62\,n = 24$$
$$62\,n = 2400$$
$$n = .387 \text{ or } 38.7\% \text{ to the nearest tenth}$$

CHECK.
$$(.387)(62) = 24.0 \text{ correct to tenths}$$

(2) 75 is 24% of what number? You know in this problem that 75 is 24% of some unknown number, which we may call n. This problem, then, says that 24% of n is 75, and this relationship can be stated as an equation.

SOLUTION.
$$.24\,n = 75$$
$$24\,n = 7500$$
$$n = 312.5$$

CHECK.
$$.24(312.5) = 75$$

(3) Bill Wilson bought a second-hand bicycle and then sold it for $6.93. He said he gained 26% on the cost. What was the cost?

The known values in this problem are the selling price of $6.93 and the gain of 26% over the cost. The unknown value in the problem is the cost. Let c be the cost.

The selling price of $6.93 is equal to the cost plus 26% of the cost. This relationship can now be stated as an equation.

SOLUTION.
$$c + .26\,c = 6.93$$
$$100\,c + 26\,c = 693$$
$$126\,c = 693$$
$$c = 5.50$$

The cost is $5.50.

CHECK.
$$.26(5.50) + 5.50 = 6.93$$
$$1.43 + 5.50 = 6.93$$

(4) An airplane engine that develops 500 horsepower at sea level has 345 horsepower at 10,000 feet. What is the per cent of decrease?

Exercises

Solve the following equations:

1. $.20\,x = 180$
2. $.35\,c = 10.50$
3. $.04\,p = 32$
4. $x + .06\,x = 3.18$
5. $c + .10\,c = 495$
6. $m - .15\,m = 21.25$
7. $x + .03\,x = 412$
8. $p + .04\,p = 520$
9. $p + .05\,p = 16.80$
10. $x - .50\,x = 18.75$
11. $x - .20\,x = 25.60$
12. $2 - 3\,x - .5\,x = 7$
13. $1.75\,x - \frac{1}{2}\,x = 1000$
14. $80 + .20\,x = x$

15. 64 is 15% of what number?

16. 17 is what per cent of 73?

17. 405 is what per cent of 225?

18. What per cent of 37 is 25?

19. 73 is 24% of what number?

Solve the following problems:

20. Find the cost of an article sold for $165 if the gain was 10% of the cost.

21. What number increased by 75% of itself equals 154?

22. After deducting 15% from the marked price of a table, a dealer sold it for $21.25. What was the marked price?

23. A dealer made a profit of $3690 this year. This is 18% less than his profit last year. Find his profit last year.

24. A number increased by 12.5% of itself equals 243. What is the number?

25. A shoe dealer wishes to make 45% on shoes. At what price must he buy them in order to sell them at $4.35 a pair?

26. A furniture dealer was forced to sell some damaged goods at 14% less than cost, and sold them for $129. How much did they cost?

27. A man sold a suit of clothes for $40.25. What per cent did he gain if the clothes cost him $25?

28. The enrollment in our junior college this year is 44, an increase of 10% over last year's enrollment. What was the enrollment last year?

29. During the last five years the house in which Mr. Wilson lives has depreciated in value 30% of the price he paid for it. If it is now worth $7700, what did he pay for it?

30. A rug cost $47.20 after a 20% discount on the original price had been deducted. What was the original price?

31. A watch cost a jeweler $28. What must the selling price be if he is to sell it at a gain of 40% of the selling price?

As early as 2000 B.C. the Babylonians had developed a system of banking. The temples of Babylon were also the banks. One ancient document reads as follows: "Two shekels of silver have been borrowed by Mas-Schamach, the son of Adadrimeni, from the sun-priestess Amat-Schamach, daughter of Warad-Enlil. He will pay the Sun-God's interest. At the time of the harvest he will pay back the sum and the interest upon it."

Interest Problems

Since interest on any principal for a year is usually a certain per cent of that principal, interest problems are special cases of problems in per cents. Many of them can be solved more easily by algebra than by arithmetic. Remember that the interest formula is

$$i = prt,$$

where i is the interest, p is the principal, r is the rate per year expressed as a decimal, and t is the time in years.

Exercises

1. A man borrowed a sum of money at 6%. Express algebraically: the interest for 2 years; the fact that the interest for three years was $48.

2. What principal must be invested at 6% to yield an annual income of $57?

3. What is the interest on P dollars at 5% for t years?

4. What is the interest for one year on $(450 - m)$ dollars at 6%? on $(1000 - n)$ dollars at $3\frac{1}{2}$%?

5. Multiply as indicated:

(*a*) $.04(350 - n)$ (*b*) $.035(900 - n)$

6. Solve the following equations:

(*a*) $.05\,n + .04(400 - n) = 18$

(*b*) $.035\,n + .06(1200 - n) = 60$

7. A man invested part of \$1000 at 4% and the remainder at 6%. The annual income on the 4% investment exceeded the annual income on the 6% investment by \$20. Find the amount invested at each rate. (Part of the solution follows.)

Amount invested at 4%	n
Amount invested at 6%	
Annual income from 4% investment	$.04\,n$
Annual income from 6% investment	

Before writing the equation, state in words the relationship between the annual incomes at the two rates.

Chapter Summary

In this chapter you have used in simple problems the same general method that is employed in all verbal problem solutions.

You have practiced the skills required for problem solving; that is, (1) to express one quantity in terms of another, (2) to formulate equations that express relations between quantities, and (3) to solve equations. You have followed the steps in the analysis and solution of a problem as they are set forth on pages 257 and 258. You have seen how the method is applied in the solution of a number of kinds of simple problems, including age problems, coin problems, time-rate-distance problems, problems involving per cents, and interest problems.

Technical problems of engineering, science, shopwork, construction, etc., have not been introduced because their unfamiliar terms and new ideas would be confusing. The important thing is the method of solving the problems, and the method you have used is the one employed in solving the mathematical problems of the world. Review the chapter and make sure that you know how to attack a verbal problem and how to carry out the various steps by which the problem can be solved.

Chapter Review

1. The difference between two numbers is 6. If 3 times the larger is subtracted from 52, the result is the same as when 56 is subtracted from 7 times the smaller. Find the numbers.

2. If I paid a bill of $2.75 with quarters and half dollars, using two more quarters than half dollars, how many of each coin did I use?

3. What must be the selling price of an article that cost $1.80, in order that the gain may be 25% of the selling price?

4. On March 1, Mrs. Jones invested $1250 in a building lot. On the fifteenth of the following September she sold the lot for $1750. What was the per cent of gain?

5. An airplane has been traveling east from Chicago, flying 150 m.p.h. for 5 hours. It is overtaken by another plane that left Chicago 2 hours after the first plane left. At what speed is the second plane traveling?

6. Two bombers set out at the same time from two bases 840 miles apart and fly toward each other with speeds of 200 m.p.h. and 220 m.p.h. respectively. After how many hours will they meet and how far has each one flown?

7. Two planes start from the same airport at the same time and fly in opposite directions. After 4 hours of flying, the planes are 1320 mi. apart. If one plane flies 30 m.p.h. faster than the other, what is the rate of the slower one?

8. Part of $1200 is invested at 5% and the remainder at 7%. The total amount of income yearly from the two investments is $67. What was the amount of each investment?

9. A 5% investment yields annually $5 less than a 4% investment. Find the amount of each if the sum of the two investments is $800.

10. The number of posts required for a fence is 85 when they are placed 18 ft. apart. How many would be needed if they were placed 12 ft. apart? (SUGGESTION. How long is the fence?)

Boeing Aircraft Company

View in the engineering room of the Boeing Aircraft Company. The engineers here have to solve many equations of such a nature that graphing is the easiest method of solution. The elements of the method are shown in this chapter.

Chapter XI

GRAPHING EQUATIONS

We graph tables of statistics to make the meaning of the numbers more easily understood. For the same reason the mathematician makes graphs of the equations with which he deals. A graph to him is a picture of an equation; from the graph he can tell at a glance how one variable in the equation changes with another. Furthermore, engineers use graphs constantly to solve equations that are difficult or impossible of solution in other ways. Aviators use graphs in solving many kinds of problems. Meteorologists and statisticians employ them constantly. Graphing of equations holds an important place in the modern mathematical and industrial world.

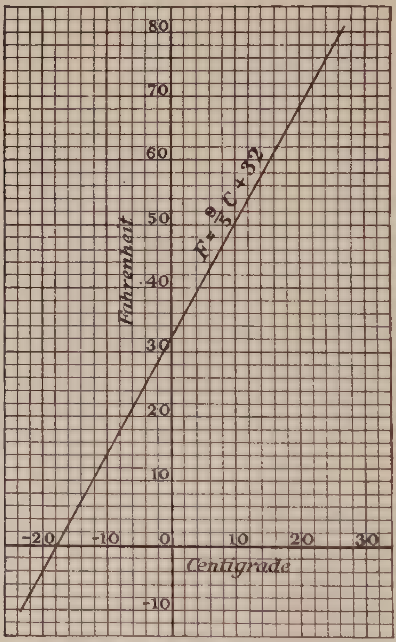
In connection with the work on formulas you have already learned how to plot points and draw graphs when all the numbers involved are positive. Now you will extend that work to include the graphing of equations in which the numbers may be negative as well as positive. At first a study of the graph of a familiar formula will acquaint you with the method used in graphing signed numbers. Following that, we shall take up the method in detail.

We shall in this chapter deal almost entirely with equations of the first degree that contain *two unknowns,* or variables. In your study you will learn the special characteristics of equations of this type, the graphs of which are always straight lines. Graphs of equations of higher degree than the first are curved lines. For example, the graphs of equations of the second degree are circles, ellipses, parabolas, and hyperbolas.

Graph Showing Relationship of Fahrenheit and Centigrade Readings

Below is a graph of the relationship between Fahrenheit (F.) and centigrade (C.) thermometer readings. The formula for the relationship is $F = \frac{9}{5}C + 32$.

(1) Where on the graph is the point corresponding to $C = 20°$, $F = 68°$? to $C = 10°$, $F = 50°$? to $C = 0°$, $F = 32°$?



(2) How can $-10°$ C. and $-20°$ C. be distinguished from $10°$ C. and $20°$ C. on the horizontal scale?

(3) How can $-10°$ F. be distinguished from $10°$ F. on the vertical scale?

(4) Where is the point corresponding to $C = -10$, $F = 14$?

(5) Where is the point corresponding to $C = -20$, $F = -4$?

(6) Refer to the graph and complete: $10°$ C. corresponds to ___?___ F.; $5°$ C. corresponds to ___?___ F.; $-10°$ C. corresponds to ___?___ F.; $54°$ F. corresponds to ___?___ C.

Equations with Two Variables

There is an important difference between the two equations $2x + 3 = 4$ and $2x + 3y = 4$. The first equation contains one unknown and there is only one value of that unknown that will satisfy the equation. The second equation contains *two unknowns*. You cannot solve this equation and find only one value of x and y that will satisfy it, for there are an unlimited number of such values. Since the values of these unknowns may vary, we call them *variables*.

(1) If I should tell you that I am thinking of two numbers whose sum is 10, could you be sure that you could guess the ones I am thinking of in 100 guesses? in 1000 guesses? I might be thinking of 1,000,010 and – 1,000,000. What is the sum of these numbers? Or I might be thinking of $4\frac{15}{71}$ and $5\frac{56}{71}$. How many pairs of numbers satisfy the equation $x + y = 10$?

(2) Which of the following pairs of numbers satisfy the equation $2x + y = 10$?

(*a*) $x = 4, y = 2$ (*d*) $x = -3, y = 16$ (*g*) (3, 4)
(*b*) $x = 6, y = -2$ (*e*) $x = 5, y = 5$ (*h*) (– 3, 4)
(*c*) $x = 0, y = 0$ (*f*) $x = 2\frac{1}{2}, y = 5$ (*i*) (5, 0)

(When a number-pair is written as (a, b), the first number is the x-value and the second is the y-value.)

(3) If in the equation $2x + y = 10$, x is 1, what is y? What is y if x is – 2? if x is 10? if x is – 3? What is x if y is 6?

From Ex. (3) you see that you can give x (or y) any value you please and then compute the corresponding value of y (or x). This shows that there are an unlimited (or infinite) number of pairs of values of x and y which satisfy the equation. Some of these pairs of values can be shown by means of a table.

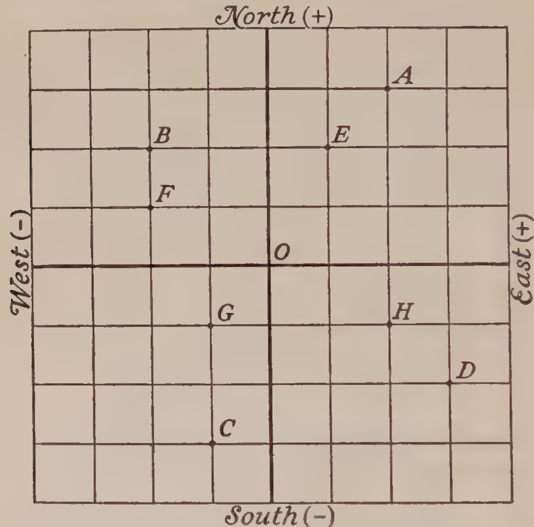
EXAMPLE. Make a table of corresponding values of the variables for the equation $2x + y = 10$. Use $x = -5, -4, -3, \cdots 4, 5$.

If x is . . .	– 5	– 4	– 3	– 2	– 1	0	1	2	3	4	5
then y is	20	18	16	14	12	10	8	6	4	2	0

From this table you get the number-pairs (– 5, 20), (– 4, 18), etc., which are the basis for graphing the equation. You will remember that in the graphing of equations numbers are represented by lengths of lines and number-pairs by points.

Locating Points on a City Map

Locating points on graph paper is like locating points on a city map in relation to some central point. Using the map following, answer the questions below it.



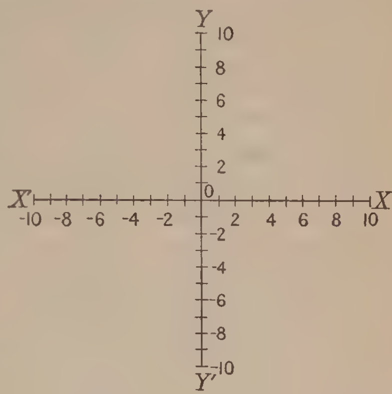
(1) If you start at O and walk two blocks east and then three blocks north, where will you be? two blocks west and two blocks north? one block west and three blocks south? three blocks east and two blocks south?

(2) Tell how to locate in this same way the points E, F, G, and H.

(3) If now we agree to use positive numbers for directions east and north and negative numbers for directions west and south, and agree also always to state the east-west direction first, we can locate A by writing the pair of numbers (2, 3); we can locate B by (– 2, 2), C by (– 1, – 3), and D by (3, – 2). Similarly indicate the location of E, F, G, and H.

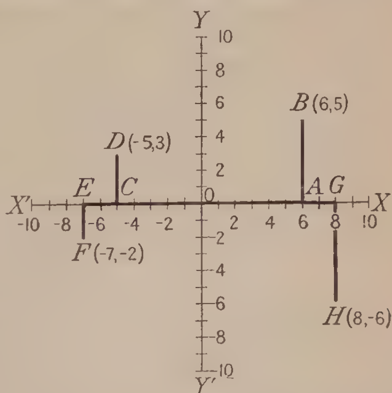
Locating Points on Graph Paper

The first step in locating, or plotting, points on graph paper is to draw two number scales at right angles to each other, just as you did when graphing formulas. This time, however, you must make provision for the negative part of the scale as shown in the figure. The horizontal line XX' (read "XX prime") is called the ***X*-axis**. The vertical line YY' is the ***Y*-axis**. As you know, O is the *origin*.



When the axes have been drawn and lettered and the scales have been numbered, you are ready to plot points. We agree that distances from the Y-axis measured along the X-axis (also distances measured parallel to the X-axis) shall be positive if they are to the right and negative if they are to the left. Similarly, distances from the X-axis measured along the Y-axis (or parallel to the Y-axis) are positive if upward and negative if downward.

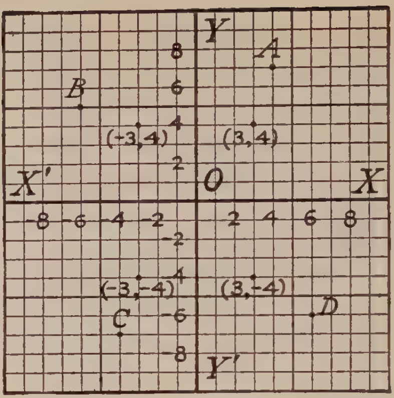
The point B on the chart at the right represents the number-pair (6, 5,) meaning $x = 6$, $y = 5$. The length OA represents 6 units and the length AB represents 5 units. The point B is 6 units to the right of the Y-axis and 5 units above the X-axis.



Describe the location of the following points in a manner similar to that employed in the preceding paragraph:

$$(-5, 3), \quad (-7, -2), \quad (8, -6).$$

These are the points D, F, and H.



Use the chart at the right in the exercises that follow.

(1) Find the points (3, 4), (– 3, 4), (– 3, – 4), and (3, – 4) on this chart and explain how they were located.

(2) What number-pairs are represented by $A, B, C,$ and D?

The x-distance of a point (its distance from the Y-axis measured along the X-axis or a line parallel to it) is often called the ***abscissa*** of the point. The y-distance of a point (its distance from the X-axis measured along the Y-axis or a line parallel to it) is the ***ordinate*** of the point. The two distances together are the ***coördinates*** of the point. Thus for the point (3, – 4), 3 is the abscissa and – 4 the ordinate; 3 and – 4 are the coördinates of the point.

(3) What are the coördinates of the origin?

(4) What is the x-distance (or abscissa) of any point on the Y-axis?

(5) What is the y-distance (or ordinate) of any point on the X-axis?

(6) What are the coördinates of points A, B, C, and D on the preceding graph chart?

Exercises

Draw axes and plot the following points on graph paper:

1. (5, 7)
2. (– 5, 7)
3. (– 5, – 7)
4. (5, – 7)
5. (0, 0)
6. (0, 3)
7. (0, – 3)
8. (3, 0)
9. (– 3, 0)
10. (8, 5)
11. (10, – 4)
12. (– 8, 7)
13. (– 6, – 3)
14. (2.5, 4)
15. (– 3.5, 7)
16. (– 1.5, – 5.5)

17. Have one of your classmates place points on a graph chart so that you can practice stating their coördinates.

18. If the x-distance of a point is 0, on what line is the point located?

19. If the y-distance of a point is 0, on what line is it located?

20. What is the x-distance of any point on a line parallel to the Y-axis 3 units to the right of it? 4 units to the left of it?

21. What is the y-distance of any point on a line parallel to the X-axis 5 units above it? 2 units below it?

Graphing an Equation of the First Degree with Two Variables[1]

An equation with two variables is of the first degree if —

1. Each variable has the exponent 1.
2. No variable is in the denominator of a fraction.
3. Not more than one variable is in any term.

The method of graphing such an equation is shown in the following example.

EXAMPLE. Draw the graph of the equation $2x + y = 6$.

(1) Solve the equation for y. Then you have $y = 6 - 2x$.

(2) Choose values for x and by substitution find the corresponding values of y.

(3) Make a table of these corresponding values of x and y:

x	-3	-2	-1	0	1	2	3
y	12	10	8	6	4	2	0

(4) Plot the points $(-3, 12)$, $(-2, 10)$, $(-1, 8)$, etc., on graph paper.

(5) Using a ruler, draw a line through these points. This line is the graph of the equation $2x + y = 6$.

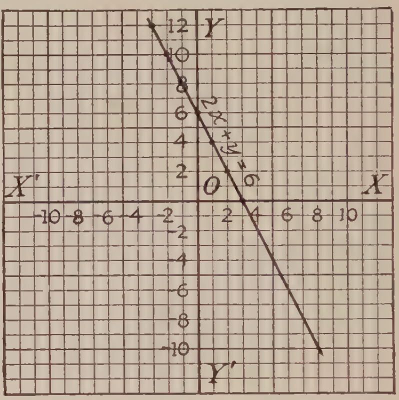
(6) Label the graph $2x + y = 6$ as shown on the next page.

[1] TO THE TEACHER. See Note 19 on page 461.

In higher mathematics it is shown that —

The graph of every equation of the first degree with two variables is a straight line.

Hence, to draw the graph of such an equation, it is necessary to plot only two points and then draw a straight line through them. You should, however, plot another point as a check. For the sake of accuracy in drawing the graph the points should not be too close together.



Equations of the first degree with two variables are often called ***linear equations*** because their graphs are straight lines.

Notice that the equation of the first degree with two variables takes the form $y = ax + b$ after you solve for y. In such a linear equation x and y are the variables and a and b are *constants* which remain the same — that is, they do not vary but remain "constant" — for a given equation.

To find the value of one variable when the other is given, it is advisable to solve for it in terms of the other. You will, therefore, find it helpful to practice solving for y in the equations given in the following Exs. 1–19.

Exercises

Solve the following equations for y:

1. $2x + y = 3$

2. $3x + y = 4$

3. $2x + 3y = 7$

4. $3x + 2y = 5$

5. Solve for y: $7x - y = 5$. (SUGGESTION. First multiply both sides of the equation by -1 in order to make the coefficient of y positive.)

Solve the following equations for y:

6. $2x - y = 3$
7. $3x - y = 4$
8. $5x - y = -6$
9. $3x - 2y = 4$
10. $2x - 3y = -7$
11. $5x - 7y = 12$

Solve the following for y, then find the value of y in each equation when x equals, respectively, $-3, -2, -1, 0, 1, 2,$ *and* 3:

12. $3x + y = 2$
13. $2x + y = 5$
14. $3x - y = -6$
15. $5x - y = -3$
16. $3x + 2y = 8$
17. $2x - 3y = 4$
18. $5x - 2y = -1$
19. $7x - 4y = 0$

Draw the graphs of the following equations after making a table containing three pairs of corresponding values of the variables:

20. $y = x + 2$
21. $y = x - 3$
22. $y = 3x - 2$
23. $y = 5x + 6$
24. $y = 3x - 4$
25. $x + y = 4$
26. $x - y = 5$
27. $3x + y = 6$
28. $2x + y = -3$
29. $4x - y = -2$
30. $y = 4x$
31. $4x - y = 0$

Draw the graphs of the following equations. Many integral values of x will give fractional values of y in these equations. Try to choose values of x that will give integral values of y.

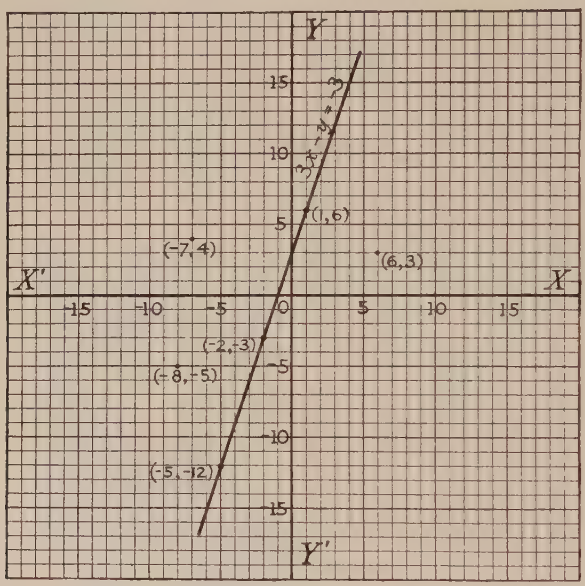
32. $2x + 3y = 12$
33. $3x - 2y = 12$
34. $4x + 5y = 14$
35. $4x - 5y = -3$

Studying the Graph of a Linear Equation

Consider the equation $3x - y = -3$ and its graph on the next page.

(1) Locate any number of points on the graph. Read their coördinates. They may be, for example, (1, 6), (− 2, − 3), (− 5, − 12), etc. Substitute the x-distance of each point for x in the equation and the y-distance for y. Is the equation satisfied by the coördinates of these points?

(2) Even if you take points on the graph with fractional and decimal coördinates, you will still find that the coördinates satisfy the equation. For instance, what is the y-distance of a point on the graph if the x-distance is 3.2? Does $x = 3.2$ and $y = 6.6$ satisfy the equation?



(3) Now take any number of points not on the graph, say (6, 3), (– 7, 4), (– 8, – 5), etc. Do the coördinates of these points satisfy the equation?

(4) Substitute any number of values for x (say 6, 3, – 4, – 7) in the equation and find the corresponding values of y. You now have several pairs of values of x and y which satisfy the equation. Plot these pairs of values on the graph paper. Do these points lie on the graph or not?

(5) Choose several pairs of x and y values which do not satisfy the equation. Plot these pairs of values on the graph paper. Do these points lie on the graph or not?

The exercises above make evident the truths about the graph of an equation given at the top of the next page.

The graph of an equation is the line, straight or curved, such that the four statements below are true:

Any point on the graph of an equation has coördinates that satisfy the equation.

Any point not on the graph has coördinates that do not satisfy the equation.

Any pair of values that satisfy the equation represent a point on the graph.

Any pair of values that do not satisfy the equation represent a point not on the graph.

Exercises

1. On what line is every point whose x-distance is 3? Draw this line. The equation of this line is $x = 3$.

Draw the graphs of the following equations:

2. $x = 5$	**4.** $x = 0$	**6.** $x = 4$
3. $x = -2$	**5.** $x = -6$	**7.** $x = 7$

8. On what line is every point whose y-distance is -2? Draw this line. The equation of this line is $y = -2$.

Draw the graphs of the following equations:

9. $y = 3$	**11.** $y = 0$	**13.** $y = 3$
10. $y = -5$	**12.** $y = -4$	**14.** $y = 6$

---

15. Find the area enclosed by the lines whose equations are $x = 3$, $y = -2$, and $y = 10 - 2x$.

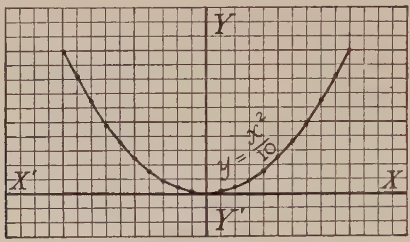
Curved Graphs

You should not be led to believe from the work in this chapter that the graphs of all equations in two variables are straight lines. This is true only when the equation is of the first degree.

(1) Make a table for the equation $y = \frac{x^2}{10}$, choosing values for x as -10, -8, etc., to 8 and 10.

x	-10	-8	-6	etc.	6	8	10
y	10	6.4	3.6		3.6	6.4	10

(2) Plot the pairs of values in your table and draw a smooth curve through the points.



Exercises

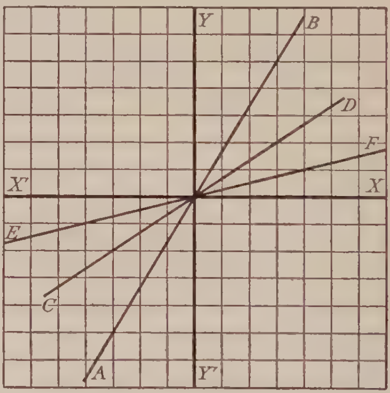
Draw the graphs of the following after making a table for each:

1. $y = \frac{x^2}{10} + 3$
2. $y = x^2$
3. $y = x^2 - 4$
4. $y = \frac{x^3}{10}$

Slopes of Straight Lines*

(1) Which line in the drawing at the right seems to you to be "steepest"? (If they represented hills, which would be the hardest to climb?)

(2) Which has the most gentle slope?



The purpose of this section is to show you a method of indicating what the *slope* of a straight line is.

In Fig. 1 on the opposite page, if you start at any point on the graph and go one unit to the right and one unit up, or two units to the right and two units up, or three units to the right and three up, etc., you will be back on the graph again. (Be sure to try this.)

* The star indicates optional material.

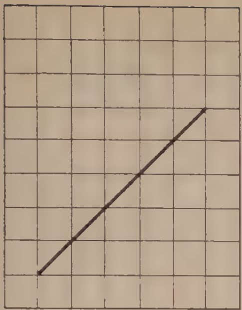


FIG. 1

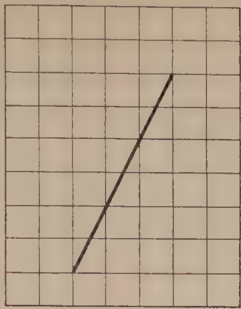


FIG. 2

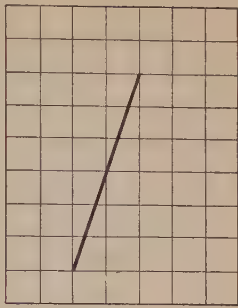


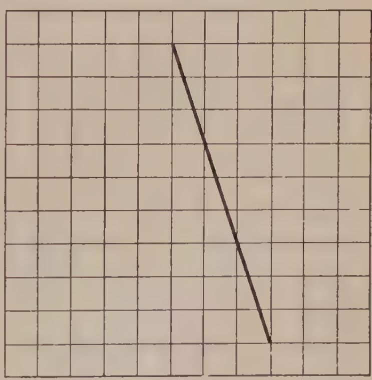
FIG. 3

In Fig. 2, however, if you go one unit to the right of any point on the graph you must go two units up, or if you go two units to the right you must go four units up, etc., before you reach the graph again.

What is the case in Fig. 3?

We can say that the *slope* of the first line is 1, that the slope of the second line is 2, and the slope of the third line is 3.

(3) On graph paper draw a line whose slope is 4. To do this, place a point anywhere on the graph paper, then go one unit to the right of it and four units up to place a second point (or go 3 units to the right and 12 units up, for greater accuracy). Draw a line through the two points.



In the figure at the right, if you start at any point on the line and go 1, 2, or 3 units to the right, you must go 3, 6, or 9 units *down* to reach the graph. Such a line has a slope of -3.

(4) Draw a line whose slope is -2. Place a point anywhere on the graph paper, go 4 units to the right of it and 8 units down, and place a second point. Draw a line through the two points.

We may have fractional slopes, also. For example, to draw a line with a slope $\frac{2}{3}$, place one point, go 3 units to the right of this point and 2 units up, and there place the second point. To draw a line with slope $-\frac{3}{2}$, place one point, then go 2 units to the right of it and 3 units down.

For all slopes: *The* denominator *indicates the number of units to the* right *of a point and the numerator indicates the number of units up or down from it.*

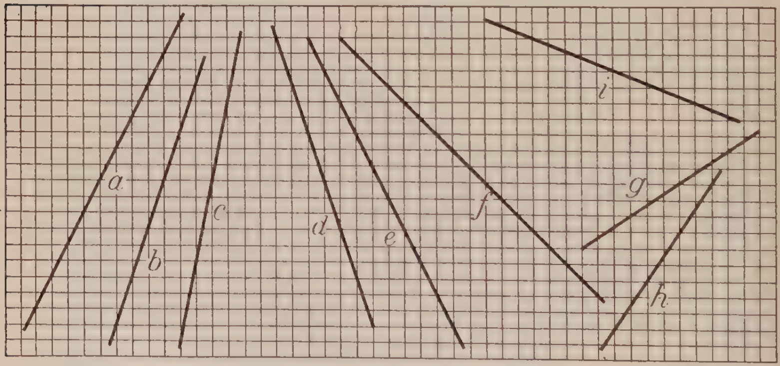
To determine the slope of a straight line, (1) start at any point on the line and go any number of units to the right, (2) determine the number of units you will have to go to reach the line again; then the slope is:

$$\frac{\textit{number of units up or down}}{\textit{number of units to right}}.$$

If you have to go *up* to meet the graph after going to the right, the slope will be *positive;* if you have to go *down* to meet the graph after going to the right, the slope will be *negative.*

Exercises

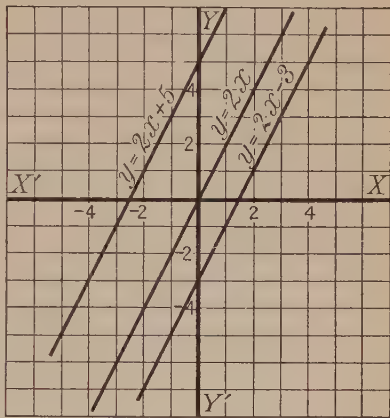
1. What is the slope of each of the lines on this chart?



2. Draw lines whose slopes are as follows: (*a*) 4, (*b*) 2, (*c*) 1, (*d*) -5, (*e*) -3, (*f*) -1, (*g*) $\frac{1}{2}$, (*h*) $\frac{1}{3}$, (*i*) $\frac{2}{3}$, (*j*) $\frac{3}{2}$, (*k*) $\frac{3}{5}$, (*l*) $\frac{5}{3}$, (*m*) $-\frac{1}{2}$, (*n*) $-\frac{1}{3}$, (*o*) $-\frac{2}{3}$, (*p*) $-\frac{3}{2}$, (*q*) $-\frac{3}{5}$, (*r*) $-\frac{5}{3}$.

Slopes and Linear Equations*

At the right you see the graphs of $y = 2x + 5$, $y = 2x$, and $y = 2x - 3$. Note from the graph that the slope of each line is 2. Note also that the coefficient of x in each equation is 2.



(1) Draw the graphs of $y = 3x + 4$, $y = 3x$, and $y = 3x - 5$ on the same axes. Note that the slope of each line is 3 and that the coefficient of x in each equation is 3.

(2) Draw the graphs of $y = 4x + 5$, $y = 4x - 1$, and $y = 4x - 7$ on the same axes. Note that the slope of each line is 4 and the coefficient of x in each equation is 4.

(3) Draw the graphs of $y = -2x + 3$, $y = -2x$, and $y = -2x - 5$ on the same axes. The slope of each line is -2 and the coefficient of x in each equation is -2.

(4) If you drew the graphs of $y = \frac{2}{3}x + 5$, $y = \frac{3}{2}x + 4$, $y = -\frac{1}{2}x + 2$, and $y = \frac{3}{4}x - 6$, you would find that the slopes were respectively $\frac{2}{3}$, $\frac{3}{2}$, $-\frac{1}{2}$, and $\frac{3}{4}$. Note the coefficients of x in the corresponding equations.

When an equation is in the form $y = ax + b$, where a can be any number except zero and b can be any number, the coefficient of x gives the slope of the graph of the equation. For example, in the graph of $y = 5x + 7$, the slope is 5. Since the coefficient of x is 5, y must increase 5 times as fast as x. In the graph of $y = -2x + 6$, the slope is -2. Since the coefficient of x is -2, y must decrease twice as fast as x increases.

As you have seen, two lines with the same slope are parallel.

(5) What is the slope of the graph of each of these equations?

(*a*) $y = 3x - 5$ (*b*) $y = 8 - 2x$ (*c*) $y = \frac{2}{3}x + 9$

Exercises

What is the slope of the graph of each of the following equations?

1. $y = 2x$

2. $y = 3x$

3. $y = 5x$

4. $y = 3x - 4$

5. $y = 6x + 2$

6. $y = 2x - 5$

7. $y = -3x$

8. $y = -2x + 3$

9. $y = -4x - 5$

10. $y = 6 - 2x$

11. $y = -7 - 3x$

12. $y = -\frac{2}{3}x$

13. $y = \frac{3}{2}x + 7$

14. $y = \frac{2}{5}x - 3$

15. $y = 8 - \frac{5}{2}x$

16. $y = \frac{1}{3}x$

17. What is the slope of the graph of $2x + 3y = 4$?

SOLUTION. You must first change this equation to the form $y = ax + b$ by solving for y.

$$2x + 3y = 4$$
$$3y = 4 - 2x$$
$$y = \frac{4 - 2x}{3}$$
$$y = \frac{4}{3} - \frac{2}{3}x$$

The slope is $-\frac{2}{3}$.

What is the slope of the graph of each of the following equations?

18. $3x + 5y = 7$

19. $7x - y = 4$

20. $5x - 3y = 8$

21. $6x + 7 = 3y$

22. $4x - 5 = -6y$

23. $7x + 2y = 4$

24. $x + \frac{1}{2}y = 2$

25. $x - \frac{1}{4}y = 0$

State how you would draw lines having the following slopes:

26. 7

27. 5

28. -2

29. -6

30. 1

31. -1

32. $-\frac{2}{3}$

33. $-\frac{3}{2}$

34. $\frac{2}{5}$

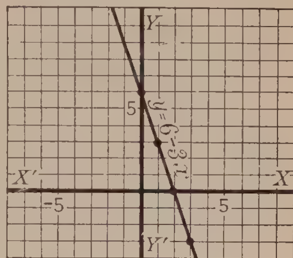
35. $\frac{5}{2}$

36. $-\frac{4}{3}$

37. $-\frac{1}{4}$

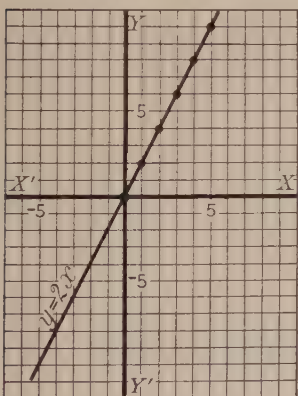
Graphing by Slopes*

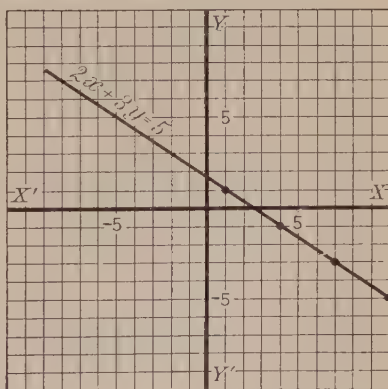
What you have learned about slopes makes it very easy for you to graph a linear equation with two variables. All you have to do is to find the coördinates of *one* point on the graph (letting $x = 0$ is usually the easiest way to get one point), and then, because you know the slope of the line, you can get as many other points as you wish.



EXAMPLE 1. Draw the graph of $y = 6 - 3x$. When $x = 0$, then $y = 6$. Plot the point (0, 6). The slope is -3. From (0, 6) go 1 unit to the right and 3 units down. Place a point. Continue this process until the first and last points are far enough apart for an accurate graph.

EXAMPLE 2. Draw the graph of $y = 2x$. If $x = 0$, then $y = 0$. Plot the point (0, 0). The slope is 2. To get another point, start at (0, 0), go 1 unit to the right and 2 units up. Place a point. Continue the process and draw the graph.





EXAMPLE 3. Draw the graph of $2x + 3y = 5$. Solve for y. $y = \frac{5 - 2x}{3} = \frac{5}{3} - \frac{2}{3}x$. When $x = 0$, $y = \frac{5}{3}$. Since y is fractional, it cannot be plotted accurately. Look for an integral value of x that will give an integral value of y. When $x = 1$, $y = 1$. Plot (1, 1). The slope is $-\frac{2}{3}$. From (1, 1) go 3 units to the

right and 2 units down. Place there another point. Continue the process until the first and the last points are far enough apart to give an accurate graph.

Exercises

1. In $y = 8x$, the y-increase is __?__ times the x-increase.

2. In the following equations, how does y increase as x increases? $y = 6x$, $y = 5x$, $y = 7x$, $y = x$.

3. What are the slopes of the lines in Ex. 2?

4. Tell how you can draw lines whose slopes are, respectively, 2, 4, 5, and 1.

5. Draw the graphs of the following equations, using the same axes for all of them: $y = 3x$, $y = 5x$, $y = 7x$, $y = x$.

6. What have all the graphs of Ex. 5 in common? How can you tell from the equations that all the graphs pass through the origin?

7. Draw the graphs of the following equations on the same axes: $y = 2x$, $y = 2x + 3$, $y = 2x - 5$.

What have all these lines in common? Could you have given your answer by looking at the equations without the graphs?

8. Draw the graphs of the following equations on the same axes: $y = 2x + 5$, $y = 4x + 5$, $y = x + 5$.

What have all these lines in common? Could you have given your answer by looking at the equations without the graphs?

9. Write four equations of the form $y = ax + b$, whose slopes are 3.

10. Draw the graph of $y = 3x - 2$. Place any point on the graph. Substitute the x-distance of this point for x and the y-distance for y in the equation. Do the coördinates of this point satisfy the equation?

Graph the following equations:

11. $y = 3x + 2$	**14.** $y = 2x - 7$	**17.** $y = x$
12. $y = 5x - 1$	**15.** $y = x - 1$	**18.** $y = 5x$
13. $y = 4x - 3$	**16.** $y = x + 2$	**19.** $y = -x$

In the following equations tell how y increases as x increases:

19. $y = 2x - 7$

20. $y = -2x + 5$

21. $y = \frac{1}{2}x - 1$

22. $y = \frac{2}{3}x + 3$

23. $y = -\frac{3}{2}x - 2$

24. $y = \frac{3}{5}x$

25. State the slope of the graph of each of the equations in Exs. 19–24.

26. Tell how you can draw graphs with the following slopes: $5, -4, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, -\frac{4}{5}, -\frac{7}{2}$.

27. Draw the graphs of the equations in Exs. 19–24.

28. What have the graphs of the following equations in common?

$$y = x, \quad y = 3x, \quad y = -2x, \quad y = \frac{3}{5}x$$

29. What have the graphs of the following equations in common?

$$y = 3x, \quad y = 3x - 2, \quad y = 3x + 5, \quad 2y = 6x + 10$$

30. What have the graphs of the following equations in common?

$$y = 2x + 7, \quad y = -3x + 7, \quad y = \frac{2}{3}x + 7, \quad 2y = 3x + 14$$

31. Draw the graphs of the following equations on the same axes: $y = \frac{3}{4}x$, $y = -\frac{3}{4}x$, $y = \frac{3}{4}x + 7$, $y = -\frac{3}{4}x - 7$.

32. Draw the graphs of the following equations on the same axes: $y = 3x - 4$, $y = -3x + 4$, $y = 3x$, $y = -3x$.

33. What is the slope of the graph of $y = 3 - 2x$? Draw the graph.

Draw the graphs of the following equations:

34. $y = 4 - 3x$

35. $y = -5 + \frac{2}{3}x$

36. $y = 2 - 5x$

37. Graph $y = x - 2$ and $y = 6 - x$ on the same axes. What are the coördinates of the point where they intersect?

Graph the following equations, first solving for y:

38. $3x + y = 1$
39. $2x - y = 5$
40. $2x + y = 9$
41. $3x - y = 6$
42. $4x + y = 5$
43. $2x + y = 2$

Draw the graphs of the following equations:

44. $2x - 3y = 4$
45. $3x - 2y = 8$
46. $5x - 3y = -4$
47. $3x - 5y = -4$

Chapter Summary

A single equation with two variables is satisfied by an unlimited number of pairs of numbers. It is impossible to show all of these in a table, since it would take forever to write down an infinite number of number-pairs. Even if you could write all of these down, you would still not have a very clear picture of the equation in question. However, it is possible to represent a limited number of number-pairs as points on a graph chart and get a very clear picture of the equation by drawing a line through these points. The points do not show the whole graph, but they do indicate the trend — that is, whether the graph is a straight line, circle, or other curved line.

The *graph of an equation* is a line, straight or curved, which contains all the points whose coördinates satisfy the equation and which contains no other points.

In this chapter you learned how to graph equations with two variables when both positive and negative numbers were involved. You did this —

(1) by drawing two number scales intersecting each other at right angles as reference lines for plotting points. The horizontal scale was called the *X-axis;* the vertical scale, the *Y-axis;* and the intersection of the two scales, the *origin.*

(2) by arranging the scales so that distances to the *right* and *up* represent *positive* numbers and to the *left* and *down* represent *negative* numbers.

(3) by finding pairs of numbers that satisfied the equations. This was done by assigning values to x, the independent variable, and determining the corresponding values of y, the dependent variable, from the equation. This gave a table of number-pairs to plot.

(4) by plotting points that represented the number-pairs in the table.

(5) by drawing a line through the points plotted to give the *graph* of the equation.

The graphs of first-degree equations with two variables are always straight lines. They are, therefore, called *linear equations*. Plotting two points is sufficient to determine the location of straight line graphs, providing that the points are far enough apart to make the drawing of the graph accurate. A third point is used as a check.

These straight line graphs are pictures of equations of the type $y = ax + b$ (after the equations are solved for y), where x and y are the *variables* that can have an infinite number of values, and a and b are *constants* whose values remain the same in any given equation.

Graphs of equations with two variables and of degree higher than the first were found to be curved lines.

If you studied the optional material, you learned (1) the meaning of the slope of a straight line, (2) that the slope of the graph of a linear equation is the same as the coefficient of x in the equation, (3) that a linear equation can be graphed, knowing one point and the slope.

You should understand the following technical terms:

X-axis	abscissa
Y-axis	ordinate
x-distance	coördinates
y-distance	linear equation
origin	variable
slope*	constant

equation of the first degree with two variables

Chapter Review

1. Give five pairs of values that satisfy the equation $2x + 3y = 10$. How many pairs of values will satisfy this equation?

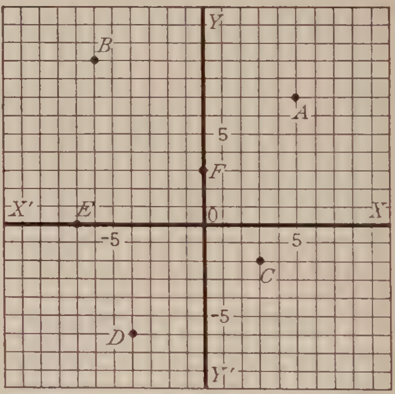
2. State without graphing whether the points (3, 5) and (2, 1) are on the graph of $2x - y = 1$. Explain.

3. On what line are all the points whose abscissas are 0? 3? -4?

4. On what line are all the points whose ordinates are 0? 5? -3?

5. Draw axes on graph paper and plot the points (7, 3), (4, -6), (-2, 3), (-5, -2), (0, 0), (0, -5), (7, 0).

6. What are the coördinates of points A, B, C, D, E, F, and O in the figure at the right?



7. Without graphing, state at what value of x the graph of $y = 2x - 3$ crosses the X-axis.

8. For what value of y does the graph of $2x + 3y = 7$ cross the Y-axis?

Solve each of the following equations for y:

9. $3x + y = 5$

10. $3x - y = 5$

11. $2x + 3y = 7$

12. $2x - 3y = -7$

Solve each of the following equations for x:

13. $3x + 2y = 4$

14. $7y - 2x = -5$

What is the value of y in each of the following equations when $x = 0$? when $x = -3$? when $x = 5$?

15. $7x + y = 9$

16. $3x - y = 8$

17. $2x - 3y = 4$

18. $2x + 3y = 6$

Draw the graphs of the following equations:

19. $x = 0$ **21.** $x = 4$ **23.** $y = -5$

20. $y = 0$ **22.** $x = -3$ **24.** $y = 6$

Draw the graphs of the following equations:

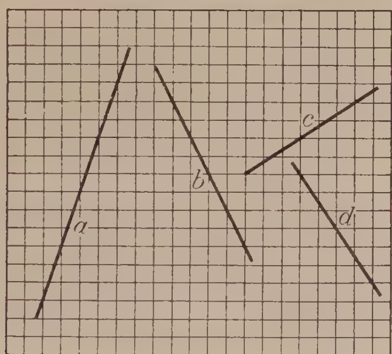
25. $y = x - 3$ **27.** $2x + y = 3$ **29.** $3x + 4y = -5$

26. $y = 3x - 7$ **28.** $5x - y = 4$ **30.** $3x - 4y = 5$

31. Draw the graph of $y = x^2 + 5$.

***32.** Straight lines with the same slope are ___?___.

***33.** What are the slopes of the lines in the chart below?



***34.** Draw lines with the following slopes: 4, -3, $\frac{3}{5}$, $-\frac{5}{2}$.

By means of slopes draw graphs of the following equations:

***35.** $y = 3x$ ***37.** $y = -\frac{3}{2}x + 3$

***36.** $y = 3x + 2$ ***38.** $3x + 4y = 12$

39. How do the graphs of second-degree equations differ from the graphs of first-degree equations?

40. Draw the graphs of $y = x^2 - 2x + 1$ and $2x - y = 2$ on the same axes. At how many points do these graphs cross each other or intersect? Read the coördinates of the points of intersection and substitute these values for the variables in both equations. What do you discover?

Maintaining Skills

(Decimals)

1. Copy the following decimals and arrange them in order of size, with the largest first: .4, .06, .15, .085.

Write the following as decimal fractions:

2. $\frac{1}{2}$ **6.** $\frac{1}{10}$ **10.** $\frac{2}{5}$ **14.** $\frac{4}{5}$ **18.** $\frac{2}{10}$

3. $\frac{1}{3}$ **7.** $\frac{1}{25}$ **11.** $\frac{3}{3}$ **15.** $\frac{3}{8}$ **19.** $\frac{3}{10}$

4. $\frac{1}{5}$ **8.** $\frac{2}{2}$ **12.** $\frac{3}{4}$ **16.** $\frac{5}{8}$ **20.** $\frac{7}{10}$

5. $\frac{1}{8}$ **9.** $\frac{2}{3}$ **13.** $\frac{3}{5}$ **17.** $\frac{7}{8}$ **21.** $\frac{9}{10}$

Write the following as common fractions:

22. .10 **25.** .50 **28.** .375 **31.** $.66\frac{2}{3}$

23. .1 **26.** .75 **29.** .875 **32.** .80

24. .5 **27.** .125 **30.** $.33\frac{1}{3}$ **33.** .625

Change to a decimal. In case there is a remainder, give the result to the nearest thousandth.

34. $\frac{3}{7}$ **36.** $\frac{1}{9}$ **38.** $\frac{5}{16}$ **40.** $\frac{11}{32}$

35. $\frac{5}{7}$ **37.** $\frac{7}{9}$ **39.** $\frac{7}{11}$ **41.** $\frac{19}{24}$

Add:

42. .7, 2.8, 9.0, 11.6 **44.** .006, .759, 8.437

43. 5.037, 0.919, 3.400 **45.** 32.03, 192.50, 73.00

Subtract the second number from the first:

46. 5.00, .83 **47.** 75.2, 3.7 **48.** 8.0, 0.8

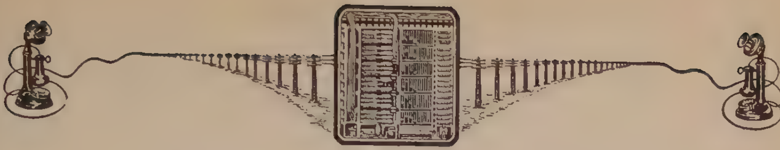
Multiply or divide as indicated:

49. 10×4.32 **53.** $4.32 \div 10$ **57.** 86.92×1.05

50. $10 \times .4$ **54.** $4.32 \div 100$ **58.** $16.675 \div 2.3$

51. 100×6.3 **55.** 1.6×0.3 **59.** $166.75 \div .023$

52. 100×9.84 **56.** $3749 \times .24$ **60.** $16675 \div .23$



Chapter XII

SOLVING PAIRS OF LINEAR EQUATIONS

It is often much easier to state the relationships in a problem by using two unknowns than it is by using only one unknown. But, as you have seen, a linear equation with two unknowns, or variables, has an unlimited number of solutions (page 275). How, then, can a problem be solved definitely by using two unknowns?

When two unknowns are used, two equations must be written with the same two variables in each. We then have a *pair of equations* and not a single equation to solve, and we must find a *common solution* for them. By a "common solution" we mean values of the variables that will satisfy both equations.

How many solutions has such a pair of linear equations, if the same values of the two unknowns must satisfy both equations? If the sum of two numbers is 10 and their difference is 2, what are the numbers? Is there more than one solution? The two equations for this problem are —

$$a + b = 10$$
$$a - b = 2$$

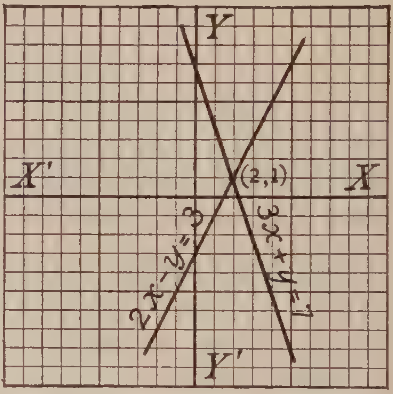
An infinite number of values of a and b will satisfy the first equation alone, or the second equation alone. What values of a and b will satisfy both equations?

The answers to these questions concerning a common solution for a pair of equations are explained in the following pages, where you will learn how to solve a pair of linear equations, first by drawing graphs and then algebraically.

Solving Pairs of Equations Graphically

Graph on the same axes $2x - y = 3$ and $3x + y = 7$. What are the coördinates of the point of intersection of these lines? The coördinates of every point on one graph satisfy its equation, and the coördinates of every point on the other graph satisfy its equation. What can you say about the coördinates of the point of intersection?

Substitute $x = 2$, $y = 1$, the coördinates of the point of intersection, for x and y in both equations to see if these values satisfy the equations.



Since the pair of numbers, $x = 2$, $y = 1$, satisfies both these equations, the equations have a common solution. Two linear equations can have only one common solution, for two straight lines can intersect at only one point.

If two equations with two variables have a common solution, they are called ***simultaneous equations.***

The steps in solving two simultaneous linear equations graphically are:

1. Graph the two equations on the same axes.

2. Find the coördinates of the point of intersection of the two graphs. This pair of values for the variables is the common solution.

3. Substitute the values found in both equations.

Simultaneous equations need not be of the first degree. Thus $y = x^2 - 2x + 1$ and $2x - y = 2$ are simultaneous equations, only one of which is linear. In many cases both equations are of higher degree than the first.

Exercises

Solve the following pairs of equations graphically:

1. $x + y = 5$
   $2x - y = 7$

2. $y = x + 4$
   $x + y = 8$

3. $y = 3x + 7$
   $2x + y = 2$

4. $2x + y = 5$
   $y = 3x$

5. $x + y = 4$
   $2x - y = 8$

6. $4x - y = 1$
   $y = 2x - 3$

7. $y = x - 1$
   $3x + y = 11$

8. $3x - y = 14$
   $2x + y = 6$

9. $y = 3x - 7$
   $y = x + 6$

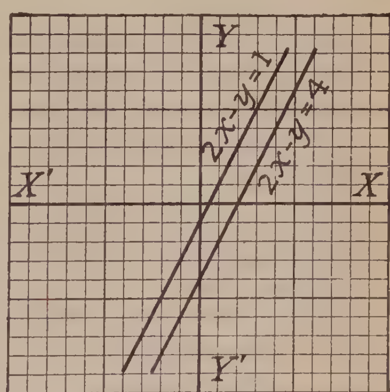
10. $2x + y = 8$
    $5x - y = 6$

11. $4x + 3y = 6$
    $2x - y = -2$

12. $3x + 4y = 10$
    $2x - y = -8$

Inconsistent and Dependent Equations

(1) Graph the equations $2x - y = 1$ and $2x - y = 4$ on the same axes. Note that the two lines are parallel and hence the equations have no common solution. It is obvious that $2x - y$ cannot be 1 and 4 at the same time; that is, for the same values of x and y.



If two equations with two variables have no common solution, they are called *inconsistent equations*.

If you have studied about slopes in the preceding chapter, you will see that both these graphs have the slope 2. If the graphs of two linear equations have the same slope, they are parallel and hence the equations are inconsistent.

(2) Graph the two equations $2x - y = 1$ and $4x - 2y = 2$. You will find that the same line is the graph of both. Hence any pair of values which is a solution of one equation is a solution of the other, or these equations have an unlimited number of common solutions. Test this statement.

Such equations are called ***dependent equations.*** One equation is dependent upon the other in the sense that it may be changed to be identical to the other equation.

If you can divide both sides of one equation of a pair by some number so as to get the other equation, you will know that the two equations are dependent.

Algebraic Solution of Simultaneous Linear Equations

The graphic method of solving simultaneous linear equations may sometimes be inaccurate because of inaccuracies in drawing. The point of intersection may have fractional coördinates which are impossible to read correctly. Then, again, the two lines may be so nearly parallel that no clear point of intersection is shown. Sometimes a more exact solution is needed and this can be obtained by the algebraic method.

We shall show you two ways of solving simultaneous linear equations algebraically. The first method is called ***elimination by substitution*** and the second is called ***elimination by addition.***

The substitution method is simple when the coefficient of either x or y in either equation is 1 or -1. When this is not so, the substitution method involves fractions and is more difficult.

Solution by Substitution

Example 1. Find the values of x and y which will satisfy both equations:

$$4x + y = 2$$
$$2x + 3y = -4$$

Solve for y in the first equation. (We choose this particular variable because here the coefficient is 1.)

$$y = 2 - 4x$$

Put this value of y in place of y in the other equation.

$$2x + 3(2 - 4x) = -4$$

You now have one equation with one unknown which can easily be solved.

$$\begin{aligned} 2x + 6 - 12x &= -4 \\ -10x &= -10 \\ x &= 1 \end{aligned}$$

Substitute $x = 1$ in the equation $y = 2 - 4x$ and get $y = -2$.

The values of x and y that satisfy both equations are $x = 1$, $y = -2$.

Check in both equations:

$$\begin{aligned} 4x + y &= 4 - 2 = 2 \\ 2x + 3y &= 2 - 6 = -4 \end{aligned}$$

EXAMPLE 2. Solve for x and y:

$$\begin{aligned} 4x - 2y &= 3 \\ x + 3y &= 13 \end{aligned}$$

Here you would first solve for x in the second equation and then substitute that value of x in the first equation. You should carry through the solution.

Exercises

Solve the following equations:

1. $x + (x + 1) = 3$

2. $(2y - 2) + 2y = 2$

3. $3x + 2(3x - 3) = 12$

4. $7x - 9(3x + 24) = -76$

5. $3(2y - 2) + 2y = -6$

6. $3x - 2(-5 - 2x) = -4$

7. $2(y + 4) + y = 5$

8. $3(7z + 7) + 5z = -31$

Solve the following pairs of simultaneous equations:

9. $\begin{aligned} 3x + y &= 7 \\ x - y &= 1 \end{aligned}$

10. $\begin{aligned} 2x + y &= 5 \\ 8x - y &= 45 \end{aligned}$

11. $\begin{aligned} x + y &= 3 \\ x - y &= 1 \end{aligned}$

12. $\begin{aligned} 3x - y &= -24 \\ 7x - 9y &= -76 \end{aligned}$

13. $y = 8 - x$
$4x - 3y = -3$

14. $3x + 2y = 16$
$7x + y = 19$

15. $2x + y = 11$
$x - y = -2$

16. $4x - 2y = 3$
$3x - y = 4$

17. $x = 3y$
$2x - 5y = 4$

18. $y = -2x$
$x + 2y = 9$

19. $x - 5y = -2$
$9x + 7y = 34$

20. $4a + b = 8$
$5a + 3b = 3$

21. $2x - 3y = 12$
$7x + y = 19$

22. $a = 3b$
$3a + 2b = 22$

23. $y = x$
$3x - 5y = 0$

24. $m + 2n = 8$
$3m - n = 3$

Solution by Addition

The addition method of solving simultaneous equations avoids fractions in cases where the substitution method would involve them. To show the new method, very simple equations that may be readily solved by the substitution method are used in Exs. (1) and (2) below, but Ex. (3) will involve fractions if the substitution method is employed.

(1) Solve the equations: $2x + y = 5$
$3x - y = 10$

If you add the left members of these equations, the result is $5x$. The y's have been eliminated. If you add the right members, the result is 15.

Now $5x = 15$, because you have added equal quantities to equal quantities and hence the results will be equal.

Therefore $x = 3$.

Substitute 3 for x in either equation and get $y = -1$.

Your solution is $x = 3$, $y = -1$.

Check by substituting 3 for x and -1 for y in both equations.

(2) Solve the equations:

$$3x + y = 11$$
$$2x + y = 8$$

If you added here, as in Ex. (1), you would eliminate neither x nor y. You can, however, eliminate y by adding after you have multiplied both sides of the second equation by -1. Do this and carry through the solution. Check your answers in both equations.

(3) Solve the equations:

$$2x + 3y = -1$$
$$5x - 2y = -12$$

Before you can eliminate y (or x) by addition, you must change the equations so that the coefficients of y (or x) are the same in absolute value but opposite in sign.

To eliminate y, first multiply both sides of the first equation by 2. Then —

$$4x + 6y = -2$$

Multiply both sides of the second equation by 3. (If the coefficient of y in the second equation had been $+2$, you would multiply by -3.) Then —

$$15x - 6y = -36$$

Adding the last two equations, you have —

$$19x = -38$$
$$x = -2$$

Substitute -2 for x in either of the original equations, and you get $y = 1$.

Your solution is $x = -2$, $y = 1$.

Check by substituting -2 for x and 1 for y in both equations.

In the examples we have eliminated y. Whenever it is more convenient, x can be eliminated instead of y.

(4) Solve:

$$2x - 4y = 6$$
$$3x - 7y = 11$$

In this case it is easier to multiply both sides of the first equation by 3 and of the second by -2. Adding then eliminates x. Complete the solution.

A summary of the elimination by addition method of solving simultaneous linear equations is as follows:

1. **Multiply, if necessary, both sides of each equation by a number that will make the coefficients of one of the unknowns have the same absolute value but opposite signs in the two equations.**

2. **Add the members of these two new equations so as to eliminate this unknown.**

3. **Solve the resulting equation for the other unknown.**

4. **Substitute your answer for this unknown in either of the given equations in order to find the value of the unknown you first eliminated.**

5. **Check by substituting your answers in both equations.**

Exercises

Solve by means of addition:

1. $3x + y = 9$
$2x - y = 1$

2. $3x + y = 10$
$2x + y = 7$

3. $5r - s = -23$
$3r - s = -15$

4. $2a + 3b = -5$
$5a + 3b = 1$

5. $3x + 2y = -7$
$5x - 2y = -1$

6. $x + 3y = 14$
$x - 2y = -1$

7. $2m + 3n = 6$
$2m - 5n = 22$

8. $2x + 3y = 8$
$3x + y = 5$

9. $2x + y = 3$
$7x - 4y = 18$

10. $4x + y = 16$
$5x - 3y = 3$

11. $5x + y = 15$
$3x + 2y = 9$

12. $7x - 5y = -2$
$8x + y = -9$

13. $4p - 2q = 20$
$p + 5q = -17$

14. $2x + 5y = 18$
$5x - y = 18$

15. $3x + 2y = 17$
$5x - 3y = 3$

16. $4a + 3b = -2$
$8a - 2b = 12$

17. $2x - 5y = 7$
$3x - 2y = -18$

18. $9c + 7d = 14$
$6c + d = 2$

19. $4x + 3y = -12$
$5x + 4y = -15$

20. $5a + 9b = 53$
$6a - 5b = 32$

21. $2x + y = 8$
$4x - y = 7$

22. $2x + 3y = 14$
$4x + 3y = 21$

23. $4a + 3b = 0$
$2a - b = -5$

24. $5x + 3y = 2$
$3x + 5y = -10$

25. $5m + 9n = 61$
$7m + 3n = 47$

26. $7x + 2y = -29$
$-9x + 5y = 7$

27. $2a + 3b = 0$
$2b + 3a = -20$

28. $5x - 6y = 1$
$3x = 2y - 33$

29. $11r = 35 + 2s$
$7r + 4s = 75$

30. $9x - 2y = 2\frac{1}{2}$
$5x - 6y = -3\frac{1}{2}$

31. $4x - 5y = -16$
$8x + 3y = -6$

32. $3x + 5y = 4\frac{1}{2}$
$9x + 2y = 7$

33. $8x + 6y = 16$
$4x - 3y = 12$

34. $5x + 4y = -\frac{1}{3}$
$7x + 2y = \frac{4}{3}$

35. $2x + 3y = 12.6$
$5x - 9y = -9.6$

36. $x + 3y = 17.3$
$2x - 7y = -31.7$

37. $ax + 3y = 8a$
$ax - y = 4a$

38. $7x + ay = 13a$
$2x - ay = 5a$

39. $6x + 5y = 46$
$10x + 3y = 66$

40. $2x + 7y = 52$
$3x - 5y = 16$

41. $2x - 7y = 8$
$4y - 9x = 19$

42. $4x - 6y = 8$
$9x + 6y = 96$

43. $2x - 3y = 8$
$3x - 7y = 7$

44. $2x + y = 7$
$2x - y = 5$

45. $3x + y = 11$
$6x + 2y = -5$

46. $2x - 3y = -14$
$3x + 7y = 48$

Solving Algebraically Problems with Two Variables

As has been stated, it is often easier to solve a problem by using two unknowns and writing two equations than to solve it by using one unknown and one equation.

EXAMPLE. Six pounds of sugar and 3 dozen eggs cost $1.68. At the same prices 4 pounds of sugar and 5 dozen eggs cost $2.44. What is the cost of each?

SOLUTION. If the cost in cents of 1 pound of sugar is x, and the cost in cents of 1 dozen eggs is y,

then 6 pounds of sugar cost $6x$ (Why?)
3 dozen eggs cost $3y$
4 pounds of sugar cost $4x$
and 5 dozen eggs cost $5y$

According to the problem:

$$6x + 3y = 168$$
$$4x + 5y = 244$$

Complete the solution by solving these equations. Check to see if your answers satisfy the given problem.

Exercises

1. Six apples and 3 pears cost 33 cents. At the same prices 3 apples and 6 pears would cost 39 cents. Find the cost of each.

2. Find two numbers whose sum is 35 and whose difference is 17.

3. The difference between two numbers is 4. Three times the larger is 2 more than 5 times the smaller. Find the numbers.

4. At a high school play, pupils paid 25 cents and adults paid 40 cents for admission. The total receipts for 90 tickets was $28.50. How many of each kind were sold?

5. The sum of two numbers is 20. Twice one of them is 3 times the other. Find the numbers.

6. The length of a rectangle is 3 inches more than the width. The perimeter is 34 inches. Find the length and the width.

7. Find two numbers such that 4 times the first number minus 5 times the second number equals 100, and 2 times the first number plus the second number equals 8.

8. For one day a contractor hired 5 men and 3 boys for $31. For another day, at the same rate, he hired 3 men and 5 boys for $25. How much did he pay each man and each boy per day?

9. John is paid $3 a week more than Helen. In 6 weeks Helen earns as much as John does in 5 weeks. What is the weekly wage of each?

10. The sum of two numbers is 16. Twice their difference increased by 3 is 7. Find the numbers.

11. Five full-fare tickets and three half-fare tickets on the railroad cost $3.25, and four full-fare and two half-fare tickets cost $2.50. How much does a full-fare ticket cost? a half-fare ticket?

Chapter Summary

If two equations with two variables have a common solution, they are called *simultaneous equations.* If two equations with two variables have no common solution, they are called *inconsistent equations* and their graphs are parallel. If every point that satisfies one equation also satisfies a second equation, the equations are said to be *dependent equations.* In this case both equations have the same graph, and the second equation can always be changed to a form identical to the first equation.

Simultaneous equations may be solved graphically by drawing their graphs on the same axes. The coördinates of the point of intersection of the graphs are the common solution. They may also be solved by algebraic methods, two of which you have studied in this chapter — *elimination by substitution* and *elimination by addition.* You should review these two methods until you understand them thoroughly.

You should understand the following technical terms:

simultaneous equations
inconsistent equations
dependent equations
elimination by substitution
elimination by addition
common solution

Chapter Review

1. How many pairs of values will satisfy both $3x + y = 4$ and $3x + y = 5$?

2. How many pairs of values will satisfy both $x + y = 5$ and $2x + 2y = 10$?

3. How many pairs of values will satisfy both $x + y = 10$ and $x - y = 2$?

4. If two equations with two variables have a common solution, they are called _?_ equations. If two linear equations with two variables have no common solution, they are called _?_ equations. Their graphs are _?_.

Solve graphically:

5. $x + y = -5$
   $3x - y = 9$

6. $3x + 2y = 11$
   $2x - 3y = 3$

Solve by substitution:

7. $y = -2x$
   $2x + 5y = 10$

8. $y = 2x + 2$
   $4x - y = 6$

9. $x + 2y = -1$
   $3x - 2y = 5$

10. $7x - 3y = -1$
    $2x - y = -1$

11. $2a + 3b = 2$
    $a - 2b = 8$

12. $p + 3q = 8$
    $2p - 10q = -40$

13. $5x + y = 0$
    $2x + 4y = -9$

14. $x + 3y = 6$
    $5x + 7y = -2$

Solve by addition:

15. $x - 2y = -13$
    $x + 2y = 3$

16. $8a + d = 4$
    $5a + d = 1$

17. $4x + 5y = 7$
    $5x + 6y = 8$

18. $3a + 5b = -19$
    $5a - 7b = 45$

* *Use slopes in making the graphs to solve the following pairs of equations:*

19. $3x + 2y = 5$
    $2x - 3y = 12$

20. $5x - 3y = -12$
    $2x + 3y = 12$

Solve the following problems, using two equations with two variables:

21. The sum of two numbers is 56; their difference is 14. Find each number.

22. Find two numbers whose sum is 40 and whose difference is 20.

23. Two girls went to the store for rice and sugar. One girl bought 3 pounds of sugar and 2 pounds of rice for 47 cents. The other girl bought 2 pounds of sugar and 2 pounds of rice for 39 cents. Find the price of a pound of each.

24. Mr. Black, a farmer, paid 10 men and 8 boys $84 for 4 days' work; later he paid 12 men and 6 boys $45 for 2 days' work. The men were paid at one uniform rate, and the boys were paid at another uniform rate. What was the daily rate of each of these boys?

25. The total value of two investments is $1100. The annual interest earned on one is 5% and on the other is 6%. The total interest is $61. How much is invested at each rate?

26. Mr. Smith wished to invest $10,000 in 3% and 4% bonds to yield an income of $330 a year. How much should he invest in each kind of bond?

27. An airliner carried on one trip 10 full-fare passengers and 2 half-fare passengers, who paid $220 for their passage. On another trip it carried 12 passengers at full fare and 1 at half fare, who paid $250. How much was the full-fare passage and how much was the half-fare?

28. The owner of a private plane purchased on a trip 30 gal. of gas and 5 qt. of oil, and on another trip he bought 25 gal. of gas and 3 qt. of oil. His first bill was $6.90 and his second bill $5.40. The prices of the oil and the gas were the same both times. What was the cost of the gas per gallon and of the oil per quart?

29. The weight of 2 light bombs and 5 heavy bombs is 5200 lb., and the weight of 4 light bombs and 3 heavy bombs is 3400 lb. Find the weight of each kind of bomb.

CUMULATIVE REVIEW[1]

1. What is meant by the *absolute value* of a signed number?

2. How many terms has the expression $2a + 3b - 4c$? How many factors has the second term?

3. What is the difference in meaning, if any, between $1a$ and a?

4. In the expression $3x^2$, what is the 3 called? what is the 2 called?

5. Find the value of each of the following: $5(2 + 7)$; $-8(5 - 2)$; $9(2 - 7)$; $-3(3 - 5)$; $2 + 5(5 + 3)$; $(2 + 5)(5 + 3)$.

6. If $x - y$ is negative, is x greater or less than y?

What is the difference in meaning between the following pairs of expressions (Exs. 7–12)?

7. $3a$, a^3

8. $2a + b$, $2(a + b)$

9. $\frac{a+b}{3}$, $a + \frac{b}{3}$

10. $(ab)^2$, ab^2

11. $x + 3(x - 2)$, $(x + 3)(x - 2)$

12. $a^2 - b^2$, $(a - b)^2$

13. Illustrate the meaning of each of the following terms in an algebraic expression: exponent, numerical coefficient, product, sum, quotient, minuend, subtrahend, numerator, denominator.

14. What is the cost of n pounds of coffee at b cents a pound?

15. If a train travels at a uniform rate of r miles an hour for k hours, how far will it go?

If p and q represent two numbers (Exs. 16–19), what will represent the following?

16. The product of the two numbers.

17. The sum of the two numbers.

18. Twice the product of the two numbers.

19. Twice the sum of the two numbers.

[1] TO THE TEACHER. See Note 18 on page 461.

20. A rectangle is 5 times as long as it is wide. If it is w feet wide, how long is it? What is the perimeter?

21. If n articles cost c cents, how much will one article cost?

If p and q represent two numbers (Exs. 22–24), what will represent —

22. 3 more than twice their product?

23. 3 more than twice their sum?

24. 3 more than twice their difference (p being larger)?

25. The sum of two numbers is s. If one is d, what is the other?

26. If the perimeter of an equilateral triangle is $6a + 9$, what is the length of one side?

27. If b is greater than a, by how much does b exceed a?

28. What number multiplied by b will give a?

29. If n represents any integer, $2n$ will be an even integer. Test this statement by using several values of n.

30. If n represents any integer, $2n + 1$ or $2n - 1$ will be an odd integer. Test this by using several values of n.

31. Write three consecutive even integers of which the middle number is $2n$.

32. Write three consecutive odd integers of which the middle number is $2n + 1$.

33. Show that the average of $n + 5$, n, and $n - 5$ is n.

34. If a man can do a piece of work in n days, what part of it can he do in one day?

Write the following statements as equations (Exs. 35–39):

35. b is 5 more than a.

36. b is 5 less than a.

37. a exceeds b by 5.

38. Twice a is 10 less than 3 times b.

39. Three times the expression $n + 5$ is 4 more than n.

40. Write as an equation: Five more than three times n is twice as much as 5 more than n.

41. A boy sells the *News* for 3 cents and the *Gazette* for 5 cents. If he sells p copies of the first and q copies of the second, how many cents should he collect?

42. If the dimensions of a rectangular box are l, w, and h, what is S, the sum of its twelve edges? What is A, the area of its six surfaces?

Subtract the lower number from the upper number:

43. $\begin{array}{r} 18 \\ -24 \\ \hline \end{array}$	**44.** $\begin{array}{r} -12 \\ -15 \\ \hline \end{array}$	**45.** $\begin{array}{r} -17 \\ -11 \\ \hline \end{array}$	**46.** $\begin{array}{r} 25 \\ 12 \\ \hline \end{array}$

Combine terms:

47. $(+9) + (-3) - (+5) - (-10)$

48. $(-8) - (+7) + (+12) - (-15)$

49. $15 - 8 + 10 + 3 - 24$

Rewrite without parentheses and combine like terms:

50. $15 - (7a - 3) + 6$

51. $18x + (9x - 7y) - 6y$

Find the following products:

52. $(+7)(+8)$

53. $(-7)(-8)$

54. $(-7)(+8)$

55. $(+7)(-8)$

56. $(+b)(-b)$

57. $(-b)(-b)$

58. $(-a)(-b)$

59. $(-a)(0)$

60. $2(x + 3)$

61. $2(x - 3)$

62. $-2(x + 3)$

63. $-2(x - 3)$

64. $(-2)(-2)(-3)(-3)$

65. $(+2)(-3)(-1)(+5)$

66. $(-3)(+1)(-1)(-5)$

67. $(+5)(-3)(+1)(+7)$

Multiply as indicated and combine similar terms:

68. $5x - 3(2x - 5)$

69. $5x + 3(2x + 5)$

70. $-5x + 3(-2x + 5)$

71. $-5x - 3(-2x + 5)$

72. What is the rule of signs in multiplying or dividing signed numbers?

Divide as indicated:

73. $\frac{10x}{5}$

74. $\frac{10x}{-5}$

75. $\frac{-10x}{-5}$

76. $\frac{-10x}{5}$

77. $\frac{10a}{a}$

78. $\frac{-10a}{a}$

79. $\frac{x^3}{x}$

80. $\frac{x^2}{x}$

81. $\frac{3x+6y}{3}$

82. $\frac{8x-12y}{4}$

83. $\frac{a^2-ab}{a}$

84. $\frac{a^2+a}{a}$

Solve and check the following equations:

85. $12x = -24$

86. $-7x = 35$

87. $-8x = -32$

88. $x - 8 = -15$

89. $x + 10 = 3$

90. $2n + 3 = n + 5$

Multiply as indicated:

91. $2(n + 3)$

92. $5(2n - 7)$

93. $(5)(2n)(-7)$

94. $(2)(n)(3)$

Perform the indicated operations:

95. $n^3 \times n^2$

96. $n \times n^2$

97. $(-n^3)(n)$

98. $(-n^4)(-n^3)$

99. $(3^2)(3^2)$

100. $(3)(3^2)$

101. $(n^2)^2$

102. $(n^2)^3$

103. $(2n)^2$

104. $(-2n)^2$

105. $(2n)^3$

106. $(-2n)^3$

107. $(2a)(3a)$

108. $(2a)(3b)$

109. $(2a)(-3a)$

110. $(-2a)(-3b)$

111. $(2x^2)^2(3x)^3$

112. $(-2x^2)^2(3x)^3$

113. Divide $x^2 + 6x + 8$ by $x + 4$.

114. Divide $x^2 - 5x + 8$ by $x - 3$.

115. Divide $x^2 - 7x + 2$ by $x + 4$.

116. The value of $x^3 + 3x - 5$ depends upon the _?_. What is the value of the expression when x is -7?

117. What is the effect upon the product abc if each factor is multiplied by 2?

118. What is the effect upon the sum $a + b + c$ if each term is multiplied by 2?

119. What is the effect upon the area of a circle if the radius is multiplied by 3? ($A = \pi r^2$.)

120. What is the area of a triangle whose base is 12 inches and whose altitude is 8 inches?

121. What is the simple interest on p dollars at 4% for t years?

122. The formula $a = p(1 + rt)$ is used to find the amount that a given principal will yield at a given rate for a given time, at simple interest. What principal must be invested at 5% for 10 years to amount to $1000?

123. Find the interest on $562 at 4% for two years.

124. Find the value of $2\pi r^2 + 2\pi rh$ when $r = 3.5$ and $h = 6$. (Use $\pi = \frac{22}{7}$.)

Find the value of the following when $a = 2$ and $b = 3$:

125. $5ab$	**127.** $7ab - 12$	**129.** $6ab - 5$
126. $3ab$	**128.** $2ab + 3$	**130.** $4ab + 2$

131. Find the value of $4a - 5b$ when $a = 7$ and $b = -5$; when $a = -3$ and $b = 4$.

Find the value of the following when $a = -5$ and $b = -2$:

132. $3a + 2b$	**134.** $a^2 + b^2$	**136.** $3(a + b)$
133. $3a - 2b$	**135.** $a^2 - b^2$	**137.** $3(a - b)$

Find the value of the following when $x = 5$ and $y = 2$:

138. $2x$ **140.** $3x$ **142.** x^2y **144.** $x - 2(y + 3)$

139. x^2 **141.** x^3 **143.** xy^2 **145.** $(x - 2)(y + 3)$

146. Find the value of $a^2 - 3ab + 4b^2$ when $a = 4$ and $b = -3$.

147. Make a table of values for x and y for the equation $y = 8 + 2x - x^2$ when $x = -4, -3, -2, -1, 0, 1, 2, 3, 4$.

Solve and check the following equations:

148. $n - 7 = -4$

149. $n + 5 = 3$

150. $7 - n = 2$

151. $n + 2 + n - 5 = 2$

152. $3n - (n + 5) = 3$

153. $n + (3 + 2n) = -3$

Solve the following equations:

154. $\frac{n}{7} = 5$

155. $n + 5 = 7$

156. $5n = 7$

157. $7n = 5$

158. $\frac{3n}{7} = 5$

159. $5n + 7n = 36$

160. $3(n + 8) = 30$

161. $2(n + 3) = 11$

Solve the following literal equations for n:

162. $n + a = b$

163. $an = b$

164. $\frac{n}{a} = b$

165. $n - a = b$

166. $2a + n = b$

167. $2(a + n) = b$

168. $\frac{3n}{a} = b + c$

169. $a^2n = b$

Solve and check:

170. $4(x - 1) = 3(x - 2) + 7$

171. $3(3x - 4) = 4(x - 5) - 32$

172. Solve the equation $5x = 0$.

Solve the following equations:

173. $3x + 8 + 2x - 2 = 31$

174. $7n = 13n + 12$

175. $3n + 3 = 17 - n$

176. $3(x - 5) = x - 4$

State definitely how the formulas in Column A have been changed to the formulas in Column B:

Column A	Column B
177. $A = lw$	**177.** $w = \frac{A}{l}$
178. $C = 2\pi r$	**178.** $r = \frac{C}{2\pi}$
179. $p = 2l + 2w$	**179.** $l = \frac{p - 2w}{2}$
180. $\frac{3}{2}l = w$	**180.** $l = \frac{2}{3}w$
181. $C = 4w + 3$	**181.** $w = \frac{C - 3}{4}$
182. $A = \frac{bh}{2}$	**182.** $h = \frac{2A}{b}$
183. $p = a + b + c$	**183.** $a = p - (b + c)$

184. The product of x and y is $p + q$. Express x in terms of p, q, and y.

185. If $A = 4x + 3y$ and $B = 4x - 3y$, what does $A + B$ equal? what does $A - B$ equal?

186. If $x^2 + 2x = 8$, what does $x^2 + 2x + 1$ equal?

187. A man and a boy together earn \$11 a day. How much does each earn if the man earns \$5 a day more than the boy?

188. A certain rectangle is 3 times as long as it is wide. If its perimeter is 64 inches, what are its dimensions?

189. C is 6 times as old as D. In 20 years C's age will be only twice D's age. What are their present ages?

190. A train traveling at the rate of 50 miles an hour covers a trip in 5 hours. How long would it take to cover the same distance if it traveled at the rate of 35 miles an hour?

191. A grocer has two kinds of tea, some worth 60 cents a pound and some worth 75 cents a pound. He has 20 pounds more of the 60-cent kind than of the 75-cent kind. How many pounds has he of each if the value of both kinds is \$45.75?

192. I bought 45 stamps for \$1.05. If part of them were 2-cent stamps and the rest were 3-cent stamps, how many of each kind did I buy?

193. John has one third as many marbles as Harry. If John buys 120 and Harry sells 23, John will then have 7 more than Harry. How many had each boy at first?

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194. How far will a bicycle wheel d feet in diameter travel in one complete revolution? How many revolutions will it make in going a mile?

195. Explain how the numbers in the following series can be represented by $\frac{1}{2^n}$, where n represents the integers 1 through 6:

$$\frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \frac{1}{32} \quad \frac{1}{64}$$

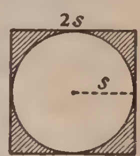
196. Explain how the numbers in the following series may be represented by $\frac{n+1}{2n+1}$ by using the integers 1 through 6 for n:

$$\frac{2}{3} \quad \frac{3}{5} \quad \frac{4}{7} \quad \frac{5}{9} \quad \frac{6}{11} \quad \frac{7}{13}$$

197. The area of a triangle is k. Its base is b. What is the altitude?

198. If the circumference of the base of a cylindrical silo is 56 feet, what is the diameter?

199. Write a formula for finding A, the area of the shaded part of the figure at the right. The side of the square is $2s$, the radius of the circle is s. Find the value of A when s is 8.



200. If $a = 2b$, what is the effect upon a if b is multiplied by 4?

201. If $a = bc$, what is the effect upon a if you multiply b by 2 and c by 3?

202. If $a = b^2$, what is the effect upon a if you multiply b by 3?

203. If $a = 2b - 3c$, does a increase or decrease when b increases? when c increases?

204. In the formula $A = \pi r^2$, what is the effect upon A if r is multiplied by 3? if r is divided by 2?

205. $V = \pi r^2 h$ is the formula for the volume of a cylinder. What is the effect upon the volume if r is doubled and h is tripled?

Solve the following literal equations for x:

206. $3x + a = x + b$

207. $5x - a = 2x + b$

208. $-x = a - b$

209. $b - 2x = x - a$

210. $2(x - a) = x + b$

211. $3(2x + a) = 4x - b$

212. Mr. Clothier made a profit of 12% on the cost of a suit. He sold it for $56. For how much did he buy it?

213. The length of a rectangle exceeds its width by 10 inches. If each dimension is increased by 5 inches, the resulting perimeter will be 128 inches. Find the area of the original rectangle.

214. If p represents the perimeter of a square each of whose sides is s, what is the perimeter in terms of p of a square each of whose sides is $3s$?

215. If A represents the area of a square each of whose sides is 2, what is the area in terms of A of a square each of whose sides is 4?

216. If the quotient is represented by q, the divisor by d, and the remainder by r, what will represent the dividend?

217. A picture twice as long as it is wide is enclosed by a frame an inch wide. The perimeter of the outer edge of the frame is 44 inches. What is the size of the picture?

CHAPTER XIII

SPECIAL WAYS OF MULTIPLYING BINOMIALS AND FACTORING

In any work you are doing, in school or out, in mathematics or in anything else, some particular bit of routine is apt to occur over and over again. In such cases the thing to do is to analyze the job to see if a way can be found to do it with fewer motions, in less time, with less energy spent, and with more efficiency. That is what is done in industry to increase production. That is what we are to do here with the method of handling binomials.

In this chapter are certain short cuts and easy ways of carrying out multiplication of binomials, which can be used frequently enough to justify a special treatment of them. We shall in this chapter make a study also of *factoring*, which is the reverse of multiplication and consists of separating an expression into the different numbers, or factors, that can be multiplied together to form it.

Multiplying Two Binomials of the Form $ax + b$ †

You have learned how to multiply two polynomials by placing one under the other. There is a much simpler method of multiplying binomials of the form $ax + b$. This method is so simple and easy that you can multiply two such binomials mentally and write down the product at sight. Examples of binomials of this form are: $2x + 3$, $3y - 4$, $-5x + 1$, $c - 2$, and $x + 5$. In the first example $a = 2$ and $b = 3$. What are a and b in each of the other examples?

† TO THE TEACHER. See Note 2 on page 458.

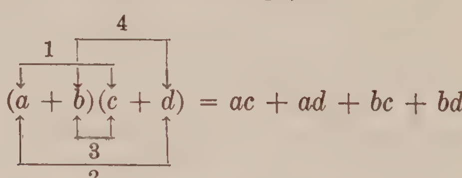
(1) In order to learn how to get the product of two binomials of the form $ax + b$, first multiply $(a + b)$ by $(c + d)$, using the method you already know of placing one under the other.

$$\begin{array}{l} c + d \\ \underline{a + b} \\ ac + ad + bc + bd \end{array}$$

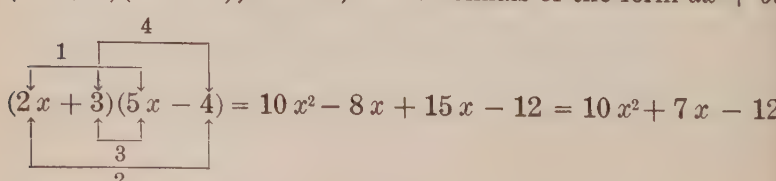
It is important for you to realize that the result of multiplying these two binomials, $(a + b)$ and $(c + d)$, gives you a pattern for multiplying *any two binomials.* a, b, c, and d are any numbers. Hence, in particular, $a + b$ could be $2x + 3$ and $c + d$ could be $5x - 4$. Here a is $2x$, b is 3, c is $5x$, and d is -4. Can you find the product of $(2x + 3)$ and $(5x - 4)$ by following the pattern without reading further?

To help in following the pattern, think of a and c as the *two first numbers,* a and d as the *two outside numbers,* b and c as the *two inside numbers,* and b and d as the *two last numbers.*

(2) Now look at the equation $(a + b)(c + d) = ac + ad + bc + bd$. Using the words in the preceding paragraph, tell what $ac + ad + bc + bd$ means. (The product of the two first numbers, plus the product of the two outside numbers, plus the product of the two inside numbers, plus the product of the two last numbers.) The algebra is much more brief than the words. This diagram will help you.



(3) Study carefully the pattern by which the multiplication is done. Then apply the pattern to the multiplication of $(2x + 3)(5x - 4)$; that is, two binomials of the form $ax + b$.



Note that the two middle terms of the product are like terms and may be combined. This is always true when the two binomials are of the form $ax + b$. At first you may need to work in two steps, but soon you should be able to combine the two middle terms mentally and write the final product at once.

(4) Practice with the following. The answers are given as a check. Cover them as you work.

(*a*) $(3x + 4)(5x - 7) = ?$	$(15x^2 - x - 28)$
(*b*) $(2x - 3)(3x + 4) = ?$	$(6x^2 - x - 12)$
(*c*) $(x - 5)(x - 2) = ?$	$(x^2 - 7x + 10)$
(*d*) $(x - 7)(x + 3) = ?$	$(x^2 - 4x - 21)$
(*e*) $(2x - 3)(2x + 3) = ?$	$(4x^2 - 9)$
(*f*) $(5x + 4)(5x - 4) = ?$	$(25x^2 - 16)$
(*g*) $(3x + 2y)(5x + 7y) = ?$	$(15x^2 + 31xy + 14y^2)$
(*h*) $(3x + 4)(2y - 3) = ?$	$(6xy - 9x + 8y - 12)$

Were you observant enough to note that the last example is not like the others? In this example the two binomials to be multiplied are not both of the form $ax + b$. The only difference in the product is that the two middle terms cannot be combined.

Exercises

Give the following products at sight:

1. $(x + 3)(x + 2)$
2. $(b + 5)(b + 3)$
3. $(x - 5)(x + 7)$
4. $(b + 2)(b - 3)$
5. $(a - 2)(a + 3)$
6. $(y - 5)(y + 1)$
7. $(b - 7)(b + 3)$
8. $(d + 4)(d - 5)$
9. $(2x + 5)(5y + 6)$
10. $(m + n)(p + q)$
11. $(3y + 2)(4y + 3)$
12. $(2x + 5)(3x + 1)$
13. $(x + 3)(2x + 7)$
14. $(p + 9)(p + 11)$
15. $(3x - 4)(2y + 3)$
16. $(m - n)(p - q)$
17. $(b + 4)(b + 4)$
18. $(n - 2)(n + 2)$

19. $(5x + 3)(x + 1)$
20. $(2f + 9)(3f + 7)$
21. $(2x - 3)(x + 3)$
22. $(5a - 4)(a + 2)$
23. $(7b + 2)(5b - 1)$
24. $(8x + 3)(3x - 4)$
25. $(10p + 3)(6p - 1)$
26. $(8r - 5)(7r + 5)$
27. $(t + 10)(t - 10)$
28. $(4x + 5)(4x - 5)$
29. $(2c - 9)(3c - 8)$
30. $(4x + 7)(5x - 3)$
31. $(3x - 5)(3x - 5)$
32. $(2x + 3)(2x + 3)$
33. $(x^2 - 4)(x^2 + 4)$
34. $(y^2 - 2)(y^2 + 2)$
35. $(2 + 3a)(3 - 5b)$
36. $(4y + 7)(3y - 5)$
37. $(7a - 2)(a + 8)$
38. $(3x - 5y)(2x + 3y)$
39. $(3c + 2d)(5c - 7d)$
40. $(2a + b)(2a + b)$
41. $(p - 5q)(3p - q)$
42. $(3x - 2y)(3x + 2y)$
43. $(2x^2 - 3y^2)(x^2 + 5y^2)$
44. $(r^3 - 2s)(r^3 + 4s)$

Perform the indicated operations and combine like terms:

45. $6x^2 - (2x + 3)(4x - 7)$. (SUGGESTION. Use a parenthesis in the second step to avoid errors in sign. Thus, $6x^2 - (8x^2 - 2x - 21)$.)
46. $a^2 - (a + 2)(a - 2)$
47. $7a - (3a + 2)(2a + 3)$
48. $4(x - 2) + (3x - 4)(x + 2)$
49. $(3n - 7)(2n + 3) + (4n - 5)(7n + 1)$
50. $(2n - 5)(5n - 4) - (3n + 7)(2n - 1)$
51. $2(3x + 1) - 5x + (3x - 2)(2x - 1)$
52. $(x + 2)(x - 3) + (x - 5)(x - 3) - 2x(x + 4)$

Solve and check the following equations:

53. $(x + 3)(x - 2) = (x - 5)(x - 7) - 2$
54. $(2n + 5)(3n - 1) = 6n^2 + 5n - 2$
55. $3(x - 2) - (2x - 1)(x + 2) = 6x - 2x^2 + 2$
56. $(2x + 3)(2x - 4) = (2x - 1)(2x - 1) - 15$

Language Used in Connection with Special Binomial Products and Factoring

Column A contains algebraic statements; Column B contains word statements or phrases translating the algebraic statements. Match the statements in the two columns.

Column A	Column B
1. $f + s$	(*a*) The difference between two numbers
2. ab	(*b*) The square of a number
3. x^2	(*c*) The sum of two numbers
4. $f - s$	(*d*) Twice the product of two numbers
5. $x^2 + y^2$	(*e*) The square of the sum of two numbers
6. $(f + s)^2$	(*f*) The sum of the squares of two numbers
7. $x^2 - y^2$	(*g*) The difference of the squares of two numbers
8. $(f - s)^2$	(*h*) The square of the sum of two numbers equals the square of the first number plus twice the product of the numbers, plus the square of the second number
9. $2\,fs$	(*i*) The product of two numbers
10. $(f + s)^2 = f^2 + 2\,fs + s^2$	(*j*) The square of the difference of two numbers
11. $(f - s)^2 = f^2 - 2\,fs + s^2$	(*k*) The square of the difference of two numbers equals the square of the first number minus twice the product of the numbers, plus the square of the second number

After matching the statements, translate each of the expressions in Column A, without referring to Column B.

Exercises

The letters a *and* b *represent two numbers. Translate the following into words:*

1. ab
2. $a - b$
3. a^2
4. $a + b$
5. $(a + b)^2$
6. $a^2 + b^2$
7. $2\,ab$
8. $(a - b)^2$
9. $(a + b)(a - b)$
10. $(a + b)^2 = a^2 + 2\,ab + b^2$
11. $(a - b)^2 = a^2 - 2\,ab + b^2$
12. $(a + b)(a - b) = a^2 - b^2$

Letting a *represent the first number and* b *the second number, write the following in algebraic symbols:*

13. The sum of the two numbers.
14. The difference of the two numbers.
15. The product of the two numbers.
16. Twice the product of the two numbers.
17. The square of the first number.
18. The square of the second number.
19. The sum of the squares of the two numbers.
20. The square of the sum of the two numbers.
21. The difference of the squares of the two numbers.
22. The square of the difference of the two numbers.
23. The square of the first minus the square of the second.
24. The sum of the two numbers multiplied by the difference of the two numbers.
25. The square of the first, plus twice the product of the two numbers, plus the square of the second number.
26. The square of the first, minus twice the product of the two numbers, plus the square of the second number.

Using $2\,a$ to represent the first number and $3\,b$ to represent the second, do Exs. 13–26 again.

Square of Any Binomial †

In learning to square the sum of any two numbers or the difference of any two numbers, we see again the power of algebra. All we have to do is square $a + b$ and $a - b$ and study the result. (a and b, of course, are any two numbers.)

(1) $(a + b)^2 = (a + b)(a + b) = ?$ $(a^2 + 2\,ab + b^2)$

(2) $(a - b)^2 = (a - b)(a - b) = ?$ $(a^2 - 2\,ab + b^2)$

(3) Calling a the first number and b the second number, what does $a^2 + 2\,ab + b^2$ say? (The square of the first number plus twice the product of the two numbers plus the square of the second number.)

(4) What does $a^2 - 2\,ab + b^2$ say? (The square of the first number minus twice the product of the two numbers plus the square of the second number.)

The rules for squaring the sum of two numbers or the difference of two numbers may be stated either in words or in algebraic symbols. Remember them in either way, whichever is easier for you. The rules are —

> **The square of the sum of any two numbers is equal to the square of the first number plus twice the product of the two numbers plus the square of the second number.**

Stated algebraically, this rule is:

$$(a + b)^2 = a^2 + 2\,ab + b^2$$

> **The square of the difference of any two numbers is equal to the square of the first number minus twice the product of the two numbers plus the square of the second number.**

Stated algebraically, this rule is:

$$(a - b)^2 = a^2 - 2\,ab + b^2$$

(5) Apply these rules to $(23)^2$. Write it as $(20 + 3)^2$. What is a? (20) What is b? (3) What is $a^2 + 2\,ab + b^2$? ($400 + 120 + 9 = 529$.) This gives you a method of squaring many numbers mentally.

(6) Square 18. $(18)^2 = (20 - 2)^2 = ?$ What is a? (20) What is b? (2) What is $a^2 - 2\,ab + b^2$? $(400 - 80 + 4 = 324)$

(7) Square $5\frac{1}{2}$. $(5\frac{1}{2})^2 = (5 + \frac{1}{2})^2 = ?$ $(25 + 5 + \frac{1}{4} = 30\frac{1}{4})$

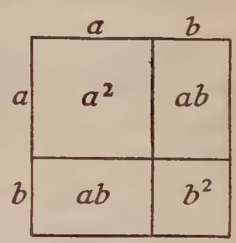
(8) Apply the rules to $(2\,x + 3)^2$. Try it first without further explanation. What is a? $(2\,x)$ What is b? (3) You now wish to write $a^2 + 2\,ab + b^2$, using a as $2\,x$ and b as 3. What is a^2? $(4\,x^2)$ What is $2\,ab$? $(12\,x)$ What is b^2? (9) Hence $(2\,x + 3)^2 = 4\,x^2 + 12\,x + 9$.

(9) Square $3\,x - 4$. $(3\,x - 4)^2 = ?$ $(9\,x^2 - 24\,x + 16)$

(10) Square as indicated: $(21)^2$, $(29)^2$, $(6\frac{1}{2})^2$, $(5\,x + 2)^2$.

Geometrical Illustration of the Square of the Sum of Two Numbers

This figure represents any square, each side of which is $a + b$ units. From the figure you see that the area of the square is $a^2 + 2\,ab + b^2$ square units. This illustrates geometrically the fact you have already learned, that $(a + b)^2 = a^2 + 2\,ab + b^2$. ($a$ and b in this illustration can be only positive numbers.)



Make a drawing that will show that $(10 + 5)^2 = 10^2 + 2 \cdot 10 \cdot 5 + 25$.

Exercises

Write the answers to the following exercises:

1. $(32)^2 = (30 + 2)^2 = ?$
2. $(28)^2 = (30 - 2)^2 = ?$
3. $(19)^2 = ?$
4. $(22)^2 = ?$
5. $(43)^2 = ?$
6. $(37)^2 = ?$
7. $(59)^2 = ?$
8. $(6\frac{1}{2})^2 = (6 + \frac{1}{2})^2 = ?$
9. $(7\frac{1}{2})^2 = ?$
10. $(8.5)^2 = (8 + .5)^2 = ?$
11. $(7.5)^2 = ?$
12. $(5.2)^2 = ?$
13. $(10\frac{1}{2})^2 = ?$
14. $(89)^2 = ?$
15. $(x + y)^2$
16. $(c + d)^2$
17. $(x + 2)^2$
18. $(n + 3)^2$
19. $(c + 4)^2$
20. $(n + 6)^2$
21. $(x + 5)^2$
22. $(x - y)^2$

23. $(c - d)^2$
24. $(x - 3)^2$
25. $(x - 2)^2$
26. $(a - 5)^2$
27. $(m - n)^2$
28. $(n - 6)^2$
29. $(2x - 3)^2$
30. $(2x - 5)^2$
31. $(2x + y)^2$
32. $(y + 10)^2$
33. $(2x - 3y)^2$
34. $(x + 8)^2$
35. $(3a - 2b)^2$
36. $(3b - 5c)^2$
37. $(3y + 4z)^2$
38. $(10a + 2)^2$
39. $(5c + 3d)^2$
40. $(2x^2 + 3y^2)^2$
41. $(5c^2 + 4y^3)^2$
42. $(a^3 - 3b^2)^2$
43. $(x^2y - 5)^2$

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44. $(x^3 - 5y^3)^2$
45. $(x + \frac{1}{2})^2$
46. $(x + \frac{1}{3})^2$
47. $(2y + \frac{1}{2})^2$
48. $(6b + \frac{1}{2})^2$
49. $(5c + \frac{1}{3})^2$
50. $(9a + \frac{1}{6})^2$
51. $\left(x - \frac{a}{2}\right)^2$
52. $(y - \frac{1}{2}p)^2$
53. $\left(x + \frac{y}{2}\right)^2$
54. $\left(\frac{x}{2} + 4\right)^2$
55. $(5b - \frac{1}{4})^2$
56. $(3y - \frac{1}{4})^2$

Product of the Sum and Difference of Two Numbers

The sum of any two numbers can be represented by $a + b$, and the difference of the same two numbers by $a - b$. Their product is indicated by $(a + b)(a - b)$, and this equals $a^2 - b^2$. (Check this statement.) The rule follows both in words and in algebraic symbols. Remember it in the way that is easier for you.

The product of the sum and difference of two numbers is equal to the square of the first number minus the square of the second number. Stated algebraically, this rule is:

$$(a + b)(a - b) = a^2 - b^2$$

Note that in the product of two such binomials there are only two terms.

(1) Practice with the following:

(*a*) $(10 + 2)(10 - 2)$
(*b*) $(20 + 1)(20 - 1)$
(*c*) $(x + 5)(x - 5)$
(*d*) $(3b + 4)(3b - 4)$
(*e*) $(x - 3y)(x + 3y)$
(*f*) $(x + y)(x - y)$
(*g*) $(x^2 - 2)(x^2 + 2)$
(*h*) $(x^2 - 2y)(x^2 + 2y)$

Exercises

Find the following products:

1. $(10 + 1)(10 - 1)$
2. $(20 + 5)(20 - 5)$
3. $51 \times 49 = (50 + 1)(50 - 1)$
4. 22×18
5. 59×61
6. 32×28
7. 99×101
8. $(p + 5)(p - 5)$
9. $(x + 8)(x - 8)$
10. $(2y + 5)(2y - 5)$
11. $(2x + 3)(2x - 3)$
12. $(3c + 5d)(3c - 5d)$
13. $(b + c)(b - c)$
14. $(c - 2)(c + 2)$
15. $(R - r)(R + r)$
16. $(2x + y)(2x - y)$
17. $(5x + 4y)(5x - 4y)$
18. $(7x - 6y)(7x + 6y)$
19. $(6x^2 - 7y)(6x^2 + 7y)$
20. $(a^2 + b^2)(a^2 - b^2)$
21. $(10 + \frac{1}{2})(10 - \frac{1}{2})$
22. $(50 + 3)(50 - 3)$
23. $(ab - c)(ab + c)$
24. $(y - x^2)(x^2 + y)$
25. $(\frac{1}{2} - 2x)(\frac{1}{2} + 2x)$
26. $(100 - 10)(100 + 10)$
27. $(2c + d)^2$
28. $(3c + 2d)(3c - 2d)$
29. $(5y - 4)(4 + 5y)$
30. $(x^2 + y^2)(x^2 + y^2)$
31. $(x^2 - y^2)(x^2 + y^2)$
32. $(x^2 - y^2)(x^2 - y^2)$
33. $(7a^2 - b)(7a^2 + b)$
34. $(2x + 3)(x + 5)$
35. $(x + y)(2x + y)$
36. 29^2
37. 32^2
38. $(a^2 + b^2)(a^2 - b^2)$

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39. $(a^4b^3 - 1)(a^4b^3 + 1)$
40. $(4c - \frac{1}{2}d)(4c + \frac{1}{2}d)$
41. $(r + .5s)(r - .5s)$
42. $(.5x - 2.5y)(.5x + 2.5y)$
43. $\left(\frac{d}{2} + 5x\right)\left(\frac{d}{2} - 5x\right)$
44. $(12x^2 + 9y)^2$
45. $(10x^2 - .1)(10x^2 + .1)$
46. $\left(\frac{x}{3} - \frac{y}{2}\right)\left(\frac{x}{3} + \frac{y}{2}\right)$

What Is Factoring?

You have been finding products when factors are given. Now you will reverse the process and find factors when products are given. We find a product by multiplying factors together. We find factors by separating a product into the numbers that were multiplied together to form it.

To factor an expression means to find two or more expressions whose product is equal to the given expression.

(1) Find factors of 10. According to the definition above, to factor 10 you must find numbers that when multiplied together give 10. In factoring integers we exclude the factor 1. Hence the factors of 10 are 2 and 5. $10 = 2 \times 5$

(2) When you know one factor of a number, how can you find the other?

(3) 3 is one factor of $3a + 3b$. How can you find the other factor? What is it? We write: $3a + 3b = 3(a + b)$.

(4) 4 is one factor of $4x + 20$. What is the other factor? $4x + 20 = 4(\quad ? \quad)$

(5) $5b^2 + 15b = 5b(b + 3)$. To check this statement, multiply $b + 3$ by $5b$ to see if you get $5b^2 + 15b$. What are the two factors of $5b^2 + 15b$ as shown here?

(6) $x^2 - 4 = (x + 2)(x - 2)$. Check this statement by multiplication. What are the two factors of $x^2 - 4$ as shown here?

When you have factored an expression, you can check your work by multiplying the factors to see if you obtain the given expression.

Removing a Common Factor

Exs. (3) and (4) in the preceding section indicate the method of factoring called *removing a common factor.*

Look at both terms of $3a + 3b$. It is obvious that 3 is a factor of both $3a$ and $3b$. Both terms have been multiplied by 3; hence 3 is one factor of the expression. Divide $3a + 3b$ by 3 to get the other factor. $3a + 3b = 3(a + b)$

(1) What is the largest factor common to all the terms of each of the following expressions?

(*a*) $5x + 20$ (*c*) $p^2 + p$ (*e*) $ax + ay - aw$

(*b*) $8y - 24$ (*d*) $3b^2 - 24b$ (*f*) $8x^3 + 12x^2 + 16x$

(2) Divide each expression in Ex. (1) by the common factor that you have found and thus find the other factor. Write the two factors of each expression.

(3) If you factor $16x + 32$, you might get $2(8x + 16)$ or $4(4x + 8)$ or $8(2x + 4)$ or $16(x + 2)$. Only the last of these sets of factors is satisfactory. You should always remove the *largest common factor;* otherwise the factoring is incomplete.

(4) In $12x^3 + 18x^2$ the largest common numerical factor is 6 and the largest literal factor is x^2. Thus $12x^3 + 18x^2 = 6x^2(2x + 3)$.

There are three steps in factoring an expression all of whose terms contain a common factor.

1. **Look at all the terms to find out what the largest common factor is. Consider literal factors as well as numerical factors. This is the first factor.**
2. **Divide the given expression by this factor.**
3. **Write the first factor followed by the second factor.**

(5) Factor $5a^2b - 5ab + 5b$, following the rules above. Check by multiplying the factors.

(6) Below are three examples of the types of error commonly made in attempting to remove the largest (or highest) common factor from a polynomial. Find the errors.

(*a*) $3a + b = 3(a + b)$

(*b*) $9a^2 + 36ab = 3(3a^2 + 12ab)$

(*c*) $5a^2b - 5ab + 5b = 5b(a^2 - a)$

Exercises

Factor the following expressions:

1. $3a + 3b$ **3.** $3m + 6n$ **5.** $8x - 40$

2. $5x - 5y$ **4.** $5a + 10$ **6.** $ax + ay$

7. $x^2 - 2xy$
8. $p^2 - 6p$
9. $25x + 15$
10. $b^3 - 2b^2$
11. $5 + 10a$
12. $3 - 9a^2$
13. $4 - 12a$
14. $p^4 - 4p$
15. $a^3 + 3a^2$
16. $x^3 + x$
17. $2a^2 - 2a$
18. $x^3 - x^2 + x$
19. $5ab + 5ac$
20. $\pi r^2 - 2\pi rh$
21. $12 + 4a^2$
22. $2ab + 4b^2$
23. $15x^3 + 5$
24. $7 - 35p^4$
25. $2 - 12a + 14a^2$
26. $8a^2 - 4a + 2$
27. $ax^2 - 3ax + 5a^2$
28. $-20x^2y - 16xy + 12xy^2$
29. $5xy - 10x^2y + 15xy^2$
30. $-6ab - 9ac - 3a$
31. $a^2b^3 - a^2b^2 + a^3b^4$
32. $3x^3y - x^2y^2 + 2xy^3$
33. $a(b - c) + d(b - c)$
34. $3x(a - b) - y(a - b)$

Factoring Trinomials of the Form $ax^2 + bx + c$

(1) Following are several examples of trinomials of the form $ax^2 + bx + c$: $2x^2 + 11x + 15$, $3x^2 + 7x - 2$, $x^2 - 2x + 1$, $x^2 - 3x + 4$, and $4y^2 + 2y - 3$. State what a, b, and c are in each case.

(2) Since $(x + 2)(x + 3) = x^2 + 5x + 6$, the factors of $x^2 + 5x + 6$ are _?_ and _?_; that is, $x^2 + 5x + 6 =$ (?) (?).

(3) Since $(2x + 5)(x + 3) = 2x^2 + 11x + 15$, the factors of $2x^2 + 11x + 15$ are _?_ and _?_; that is, $2x^2 + 11x + 15$ = (?)(?).

If a trionomial of this form can be factored, the two factors are binomials. The factoring is done by trial.

EXAMPLE 1. Factor $x^2 - 5x + 6$. When $a = 1$ as in this case, finding the factors is simple. We look for two numbers whose product is 6. They may be $+1$ and $+6$, -1 and -6, $+2$ and $+3$, or -2 and -3. At the same time, the sum of the two numbers must be -5. We know then that the numbers must be -2 and -3. Hence $x^2 - 5x + 6 = (x - 2)(x - 3)$. To check, multiply the two factors.

EXAMPLE 2. Factor $x^2 - 3x - 40$. We look for two numbers whose product is -40 and whose sum is -3. We know that they must be opposite in sign because the product is negative. We know that the larger is negative because the sum is negative. They are obviously -8 and 5. Hence $x^2 - 3x - 40 = (x - 8)(x + 5)$.

EXAMPLE 3. Factor $12x^2 - 5x - 2$. The factors of $12x^2$ may be x and $12x$, $2x$ and $6x$, $3x$ and $4x$, or the negatives of these numbers. The factors of -2 may be $+1$ and -2, or -1 and $+2$. We have to try every possible combination until the product of the factors is the right expression or until we have tried all of them and know that the expression is prime (not factorable).

Try $(x - 2)(12x + 1)$. This gives $12x^2 - 23x - 2$ and is not correct.

Try $(2x - 1)(6x + 2)$. Here the second factor has a common factor 2. The original trinomial has no such common factor; so we know that this is not correct even without multiplying.

Try $(3x - 1)(4x + 2)$. This is not correct for the same reason as the preceding one.

Try $(3x + 2)(4x - 1)$. This gives $12x^2 + 5x - 2$, which is correct except for the sign of the middle term. Interchange the signs of the second term in each factor and you have $12x^2 - 5x - 2 = (3x - 2)(4x + 1)$. This is correct.

(4) Determine whether the following expressions have been factored correctly:

(a) $3p^2 + 7p + 2 = (3p + 1)(p + 2)$
(b) $3p^2 - 7p + 2 = (3p - 1)(p - 2)$
(c) $3p^2 - 5p - 2 = (3p + 1)(p - 2)$
(d) $3p^2 + 5p - 2 = (3p - 1)(p + 2)$
(e) $6a^2 - 13a + 6 = (2a - 3)(3a - 2)$
(f) $2x^2 + 5x - 12 = (2x + 3)(x - 4)$
(g) $30a^2 - 61a + 30 = (6a + 5)(5a + 6)$
(h) $2x^2y^2 + xy - 15 = (xy + 3)(2xy - 5)$

Exercises

Factor the following expressions. (Some of them may be prime.)

1. $x^2 + 4x + 3$
2. $n^2 + 5n + 6$
3. $n^2 + 4n + 7$
4. $a^2 - 6a + 8$
5. $a^2 - 7a + 12$
6. $b^2 - 3b - 10$
7. $x^2 - 5x + 8$
8. $p^2 - p - 6$
9. $p^2 + 2p - 3$
10. $k^2 + 5k - 14$
11. $x^2 + 2x + 1$
12. $x^2 + 2x$
13. $3a^2 - 6b^2$
14. $n^2 + 6n + 9$
15. $b^2 - 10b + 25$
16. $b^2 - 10b + b^3$
17. $x^3 + x^2 - x$
18. $p^2 - 8p + 12$
19. $a^2 - 13a + 24$
20. $s^2 + 7s + 12$
21. $r^2 - r + 12$
22. $y^2 - 4y - 12$
23. $x^2 + 4x - 12$
24. $n^2 + 13n + 12$
25. $n^2 - 11n - 12$
26. $a^2 + 11a + 24$
27. $a^2 - 5a - 24$
28. $a^2 + 2a - 24$
29. $a^2 - 10a + 24$
30. $b^3 - 9b^2 + 24b$
31. $a^2 - 14ab + 24b^2$
32. $x^2 - 2xy - 35y^2$
33. $x^2 - 12xy + 35y^2$
34. $p^2 + p - 6$
35. $x^2 + 5xy + 6y^2$
36. $a^2 - 12ab + 36b^2$
37. $p^2 + 13pq + 36q^2$
38. $16c^2 - 8cd$
39. $2t^2 - 3t + 1$
40. $5r^2 + 7r + 2$
41. $2x^2 - x - 1$
42. $3y^2 + y - 2$
43. $x^4 - 7x^2 + 10$
44. $x^4 + 2x^2y^2 + y^4$
45. $3x^2 + 7x + 2$
46. $3x^2 - 5x - 2$
47. $5x^2 - 9x - 2$
48. $7x^2 + 22x + 3$
49. $2x^2 + 13x - 7$
50. $2y^2 - 5y + 3$

51. $3a^2 - 10a + 8$
52. $6x^2 + 7x + 2$
53. $4y^2 - 10y + 6$
54. $6x^2 + 5x - 6$
55. $6x^2 + 13x + 18$
56. $10r^2 + r - 2$
57. $12x^2 - 29xy + 14y^2$
58. $6x^2 + x - 1$
59. $4x^2 - 20x + 25$
60. $8y^2 + 2y - 1$
61. $15a^2 - 8a + 1$
62. $6x^2 - 13x + 6$
63. $6x^2 + 13xy + 6y^2$
64. $6x^2 - 35xy - 6y^2$
65. $6x^2 - 5xy - 6y^2$
66. $6x^2 - 16xy - 6y^2$

Factoring the Difference of Two Squares

You know that $(a + b)(a - b) = a^2 - b^2$, which is the difference of two squares. Consequently, you know that the factors of $a^2 - b^2$ are $a + b$ and $a - b$. First you must learn to recognize the difference of two squares.

(1) Which of the following are squares of numbers? If they are squares, of what numbers are they squares? (A number with an exponent is a square if the exponent is even.)

(*a*) x^2 (*d*) 25 (*g*) $3x^2$ (*j*) $16p$ (*m*) x^6
(*b*) $4a^2$ (*e*) 16 (*h*) $9n^2$ (*k*) 1 (*n*) $5x^6$
(*c*) 9 (*f*) x^3 (*i*) $49p^2$ (*l*) x^4 (*o*) x^7

(2) Which of the following are the difference of two squares? Explain how you recognize them.

(*a*) $x^2 - 9$ (*d*) $3x^2 - 25$ (*g*) $49p^2 - 16p$
(*b*) $4a^2 - 25$ (*e*) $a^2 - b^2$ (*h*) $x^4 - 9b^2$
(*c*) $x^2 + 16$ (*f*) $4n^2 - 25$ (*i*) $9a^3 - 16b^2$

EXAMPLE. Factor $25x^2 - 9y^2$.

Note that $25x^2$ is the square of $5x$, and $9y^2$ is the square of $3y$. The expression $25x^2 - 9y^2$ is therefore the difference of two squares. The factors are $5x + 3y$ and $5x - 3y$.

$$25x^2 - 9y^2 = (5x + 3y)(5x - 3y)$$

The difference of the squares of two numbers is equal to the sum of the two numbers times the difference of the two numbers; or

$$a^2 - b^2 = (a + b)(a - b)$$

Exercises

Factor the following binomials:

1. $x^2 - 64$
2. $x^2 - y^2$
3. $m^2 - n^2$
4. $1 - x^2$
5. $3x + 3y$
6. $a^4 - a^2$
7. $4a^2 - 81b^2$
8. $x^4 - y^2$
9. $p^2 + 2p$
10. $9m^2 - 100n^2$
11. $\pi a + \pi b$
12. $25 - n^2$
13. $4a^2 - 9$
14. $a^2 - 4b^2$
15. $121a^2b^2 - 1$
16. $169 - 25a^2$
17. $a^2b^2 - c^2$
18. $64a^2 - 9b^2$
19. $R^2 - r^2$
20. $a^3 - a^2$

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21. $b^2 - \frac{4}{49}$
22. $\frac{16}{25} - \frac{4}{9}t^2$
23. $.25x^2 - 1$
24. $.36a^2 - .49b^2$
25. $a^{2b} - c^{2b}$
26. $4r^{2a} - 25s^{4b}$
27. $18^2 - 12^2$
28. $64^2 - 36^2$

Factoring Trinomial Squares

A trinomial square is a trinomial which is the square of a binomial. Thus $9x^2 + 24x + 16$ is a trinomial square because it is $(3x + 4)^2$.

To factor such a trinomial is merely a matter of recognizing it as a trinomial square. Let us analyze the two trinomial squares $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$, which are the squares of $a + b$ and $a - b$.

We see that the first and last terms are the squares of a and b and that they are positive and the middle term is twice the product of a and b. a is the square root of the first term a^2, and b is the square root of the last term b^2. Hence —

> **A *trinomial* is a *square* when two of its terms are squares of numbers and positive and the remaining term is twice the product of the two numbers.**

EXAMPLE. $4x^2 - 12xy + 9y^2$ is a trinomial square. The terms $4x^2$ and $9y^2$ are positive and are squares of the numbers $2x$ or $-2x$ and $3y$ or $-3y$. $-12xy$ is twice the product of $2x$ and $-3y$ or of $-2x$ and $+3y$.

Hence $\quad 4x^2 - 12xy + 9y^2 = (2x - 3y)^2$.

Show that $4x^2 - 12xy + 9y^2$ also equals $(-2x + 3y)^2$.

Study the following exercises:

(a) $x^2 - 14x + 49 = (x - 7)^2$
(b) $4x^2 + 20x + 25 = (2x + 5)^2$
(c) $a^2 - 2ab + b^2 = (a - b)^2$
(d) $n^2 - 13n + 36 = (n - 9)(n - 4)$

In (a), (b), and (c) two of the terms are __?__ of numbers and positive. The remaining term is twice the __?__ of the numbers. These trinomials are squares.

In (d) two of the terms are squares of numbers and positive. The remaining term is not twice the __?__ of the numbers. This trinomial is not a square.

To factor a trinomial square, all you have to do is to take the square root of the first and last terms and connect them with the plus or the minus sign as the case may be. Thus —

$$25x^2 + 30xy + 9y^2 = (5x + 3y)^2$$
$$25x^2 - 30xy + 9y^2 = (5x - 3y)^2$$

Exercises

Which of the following are trinomial squares and which are not? Be ready to state orally how you tested each one.

1. $n^2 + 14n + 49$
2. $x^2 + 6x + 9$
3. $x^2 - 16xy + 64y^2$
4. $x^2 - 5x + 25$
5. $p^2 - 10pq + 25q^2$
6. $a^2 + 6a + 8$
7. $n^2 - 13n + 36$
8. $a^2 - 20a + 64$
9. $4x^2 - 12x + 9$
10. $4x^2 + 12x + 9$
11. $b^2 + 9 - 6b$
12. $n^2 - 9n - 36$
13. $y^2 + 10y + 16$
14. $x^2 - 4xy + 4y^2$
15. $25a^2 + 70ab + 49b^2$
16. $4x^2 - 20xy + 25y$

Finding Prime Factors

An integer or an algebraic expression is *prime* if it cannot be factored; that is, if it cannot be divided evenly except by itself and 1. The expression $5x - 20$ is not prime because its terms contain a common factor 5. Neither is $a^2 - b^2$ prime, for it is the product of $(a + b)$ and $(a - b)$. The following are examples of prime numbers: 3, 5, 7, 11, 13, $a^2 + b^2$, $n^2 + 7n + 4$.

When you factor an expression, you should make sure that all the factors are *prime*.

Two students factored $3x^2 + 24x + 45$ as follows:

(*a*) $3x^2 + 24x + 45 = (3x + 15)(x + 3)$
(*b*) $3x^2 + 24x + 45 = (x + 5)(3x + 9)$

They thought their answers were correct because each set of factors when multiplied gave the original expression. Note, however, that each term of the factors $3x + 15$ and $3x + 9$ has a common factor 3. They are not prime factors. The factoring is therefore incomplete.

To factor this expression completely, you should note first that the three terms contain the common factor 3 and rewrite it as $3(x^2 + 8x + 15)$. The second factor, $x^2 + 8x + 15$, can now be factored easily. The factors are $x + 3$ and $x + 5$. Hence —

$$3x^2 + 24x + 45 = 3(x + 3)(x + 5).$$

In factoring any expression, always look for a common factor before attempting any other method of factoring.

Factor $x^4 - y^4$. This is the difference of two squares, and the factors are $x^2 + y^2$ and $x^2 - y^2$. But the second of these factors can be factored further into $(x + y)(x - y)$. Hence —

$$x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y).$$

After you have factored an expression, you should look at each of the factors to see if any of them can be factored again.

Exercises

Find the prime factors of these expressions. (Some of them are already prime.)

1. $2x^2 - 2y^2$
2. $3a^2 - 3b^2$
3. $2p^2 - 8q^2$
4. $3a^2 - 6b^2$
5. $p^3 - p$
6. $y^2 + 10$
7. $2a^2 + 6a + 8$
8. $3a^2 - 3a - 18$
9. $3y^2 - 3y - 36$
10. $4a^2 - 36b^2$
11. $2x^2 - 8$
12. $2x^2 + 20x + 50$
13. $n^2 - 7n + 25$
14. $2x^2 - 12x + 18$
15. $3x^2 - 108$
16. $4x^2 - 24x + 36$
17. $a^3 - ab^2$
18. $a^3 + ab^2$
19. $2x^3 - 8x$
20. $5x^2 - 125$
21. $5x^2 + 125$
22. $c^4 - a^2$
23. $1 - a^2$
24. $1 + a^2$
25. $8a^3 - 2a$
26. $2a + ab$
27. $a^4 - b^4$
28. $p^4 - 81$
29. $4x^2 - 12x - 40$
30. $2x^2 + x - 1$
31. $3x^2 - 7x + 2$
32. $x^4 - 16$
33. $x^4 + 16$
34. $b^4 - a^4$
35. $a^2 + 16 - 8a$
36. $1 - a^4$
37. $1 - 9a$
38. $1 - 9a^2$
39. $x^2 - \frac{1}{4}$
40. $x^2 + 1$
41. $x^2 - 1$
42. $2x + 2y$
43. $2xy + y$
44. $x^3 - 4xy^2$
45. $8y^2 - 2x^2$
46. $4a^2 + 8ab + 9a$
47. $a^3 + 2a^2 + a$
48. $x^4 - 6x^3 + 9x^2$
49. $a - 16ax^4$
50. $b^4 + b^3 + b^2$
51. $a^3x^3 + 7a^3x^2 + 12a^3$

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52. $12x^2 - 26x + 12$
53. $24x^2 + 42x - 45$
54. $x^{2a} + 3x^a + 2$
55. $x^8 - y^8$

Practice in Multiplying and in Factoring

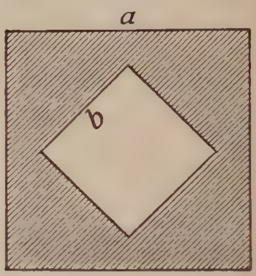
Cover Column B and factor the expressions in Column A. The answers are in Column B. Then cover Column A and multiply the factors in Column B.

Column A	Column B
1. $2a^2 + 14a + 24$	1. $2(a+3)(a+4)$
2. $5y^2 - 45$	2. $5(y+3)(y-3)$
3. $st^2 - st - 20s$	3. $s(t-5)(t+4)$
4. $7a^2 - 14a - 105$	4. $7(a-5)(a+3)$
5. $3x^2 + 12x + 45$	5. $3(x^2 + 4x + 15)$
6. $x^2 - 6x + 9$	6. $(x-3)^2$
7. $6t^2 - 15t^3$	7. $3t^2(2-5t)$
8. $2 - 128t^2$	8. $2(1+8t)(1-8t)$
9. $ab^2 - ab - 72a$	9. $a(b-9)(b+8)$
10. $n^2 + 5n + 7$	10. Prime
11. $3a^2 + a - 2$	11. $(3a-2)(a+1)$
12. $2a^2 - 5a + 3$	12. $(2a-3)(a-1)$
13. $a^2 + 25$	13. Prime
14. $q^2 - 12q - 28$	14. $(q-14)(q+2)$
15. $2x^2 - 14x + 24$	15. $2(x-3)(x-4)$
16. $y^4 - 6y^2 - 16$	16. $(y^2-8)(y^2+2)$
17. $49c^2 + 70c + 25$	17. $(7c+5)^2$
18. $5a^2 - 80$	18. $5(a+4)(a-4)$
19. $x^3 - x$	19. $x(x+1)(x-1)$
20. $a^4 - 16$	20. $(a+2)(a-2)(a^2+4)$
21. $1 - 4y^2$	21. $(1+2y)(1-2y)$
22. $x^2 + 4y^2$	22. Prime
23. $3a^2 - a - 2$	23. $(3a+2)(a-1)$

Using Factoring in Computation

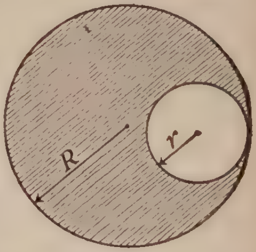
1. Find the value of $a^2 + 2ab + b^2$ when $a = 7$ and $b = 4$. Factor the expression and then find the value, using the factors. Which is the easier computation?

2. Find the value of $a^2 - b^2$ when $a = 5.7$ and $b = 2.3$. Find the value of the expression by using its factors. Which is the easier computation?



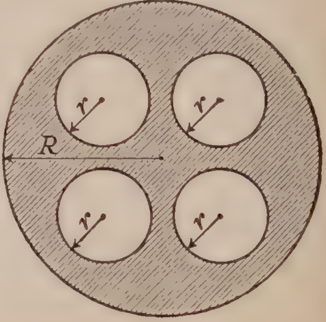
3. This figure shows a large square whose side is a units. Within the large square is a small square (in any position) whose side is b units. What is the area of the large square? What is the area of the small square? What is the area of the shaded surface?

4. Using the formula you have just written for Ex. 3, find the area of the shaded surface when $a = 28$ and $b = 22$. Then factor the expression and find the area, using the factors. Which is the easier computation?



5. A small circle whose radius is r units is drawn within a large circle whose radius is R units. What is the area of the large circle? What is the area of the small circle? What is the area of the shaded part of the figure?

6. Find the area of the shaded part of the figure for Ex. 5 when $R = 12$ and $r = 8$. Is it easier to use the formula you derived first or its factors? (What are the factors?)



7. If four circular holes (radius r) are cut into a large circular plate of radius R, what is the area of the remaining surface?

8. Find the area of the shaded part of the figure for Ex. 7, when $R = 32$ and $r = 6$, using either the formula you have written or its factors, whichever you think is easier.

Chapter Summary

The multiplying of certain special binomials occurs so frequently in algebra that it is profitable to perform these operations by time-saving rules. These rules enable you to multiply mentally: (1) two binomials of the type $ax + b$, (2) two identical binomials (the square of a binomial), and (3) the product of the sum and the difference of two numbers.

Multiplying these pairs of binomials gave you the basis for learning the reverse process of *factoring* the products of the type you derived in (1), (2), and (3) above; that is, (1) factoring trinomials of the form $ax^2 + bx + c$, (2) trinomial squares, and (3) the difference of two squares. Factors common to all terms of an expression should be removed before any other factoring is done.

You should review the rules learned in this chapter and be sure that you know how to apply them.

You should be familiar with the following terms:

Binomial of the form $ax + b$
Square of a binomial
Product of the sum and the difference of two numbers
Factoring
Prime factor
Trinomial of the form $ax^2 + bx + c$
Trinomial squares
Difference of two squares

Chapter Review

What is the largest common factor in the terms of each of the following expressions?

1. $3a - 6b$
2. $x^4 + 2x^2 + 6x$
3. $5a^2 - 20ab$
4. $6a^2b^3 + 9ab^4$

5. The largest common factor of the terms in the expression $8a^2b - 4ab^3 + 2ab$ is $2ab$. How do you find the other factor? What is it?

6. Is the following statement correct? $(ax + b)(cx + d) = acx^2 + (bc + ad)x + bd$. Explain.

Which of the following are trinomial squares, which are the difference of two squares, and which are neither?

7. $x^2 - 4$
8. $4a^2 + 4a + 1$
9. $a^2 + 16$
10. $a^2 + 6a - 9$
11. $4x^2 + 13xy + 36y^2$
12. $b^4 - 64y^2$
13. $16x^2 + 40xy + 25y^2$
14. $4x^2 + 12xy + 9$

State in algebraic symbols the rule for:

15. Squaring the sum of two numbers,

16. Squaring the difference of two numbers, and

17. Multiplying the sum of two numbers by the difference of the same two numbers.

State in words:

18. $(a + b)^2 = a^2 + 2ab + b^2$
19. $(a - b)^2 = a^2 - 2ab + b^2$
20. $(a + b)(a - b) = a^2 - b^2$

Write the following products at sight:

21. $(20 + 1)(20 - 1)$
22. $(50 + 2)^2$
23. $(7\frac{1}{2})^2$
24. $(38)^2$
25. $(38)(42)$
26. $(x + 2)(x - 2)$
27. $(3x - 4)(3x + 4)$
28. $(3x + 1)(x + 2)$
29. $(x - 4)^2$
30. $(3x + 5)^2$
31. $(1.02)^2$
32. $(3x + 1)(x - 2)$
33. $(7x - 2)(7x + 2)$
34. $(2x - 5)^2$
35. $(x - 12)(x + 11)$
36. $(3x + 2y)(2x + 3y)$
37. $(7.5)^2$
38. $(5p - 8q)(5p - 7q)$
39. $(a^3 - b^2)(a^3 + b^2)$
40. $(3x + 2)(2y - 3)$
41. $2(x - 1)(x + 1)$
42. $3(a + 2)^2$
43. $(x^2 + y^2)(x + y)(x - y)$
44. $5(x - 2)(x - 3)$

Multiply as indicated and combine like terms:

45. $3(x - 1) + (2x - 3)(x + 2)$
46. $(n + 1)(n - 1) + (4n - 5)(6n + 1)$
47. $(a - 3b)^2 - (a - 2b)(a + 2b)$
48. $3(2b - 4)(5b + 2)$
49. $2(3x + 4)(2x + 3)$
50. $-5(y - 3)(y + 3)$
51. $(a + b)^2 + (a - b)^2$
52. $(a + b)^2 - (a - b)^2$
53. $8(3x + 5)^2$
54. $-2(2x - 7)^2$
55. $4(2x - 3)(2x + 3) + 5(3x - 1)^2$
56. $4(2x - 3)(2x + 3) - 5(3x - 1)^2$

Find the values of each of the following expressions for the values of the letters given:

57. $4(2n + 5)(3n - 1)$; $n = 2$ and $n = -3$
58. $6(3n - 2)^2$; $n = -1$ and $n = 4$
59. $-5(n + 3)(n - 3)$; $n = 5$, $n = -7$, and $n = 3$
60. $2(3x - 2)(3x + 1) - 3(5x - 6)^2$; $x = 3$ and $x = -2$

Solve and check the following equations:

61. $(3n - 4)(3n + 4) = 9n^2 + 6n - 4$
62. $2(n + 2)(n - 3) + 3n^2 = 5n^2 + 2$
63. $(2n + 3)(2n - 3) + 5n = 4n^2 + 1$
64. $3(x - 1) + (2x - 3)(x + 2) = 7 + 2x^2$

Find the prime factors of each of the following expressions:

65. $x^2 + 9x + 20$
66. $t^2 - 3t - 40$
67. $64m^2 - 25y^2$
68. $2a^2 + 2a - 144$
69. $a^2 + 6a + 9$
70. $3x^3 + 6x^2 + 9x$
71. $a^3 - ab^2$
72. $y - 4y^3$
73. $3x^2 + 8x + 5$
74. $2x^2 - 5x + 3$
75. $x^4 - 16y^4$
76. $ax^4 - ay^4$

"If we could only realize the many hundreds and thousands of hours that are lost to production activity by workers in such mundane activities as converting fractions to decimals, our complacency and smugness would quickly vanish. Whenever the person doing the work does not have the needed mathematical knowledge, delay is occasioned by his having to bring the problem to a superior."

J. KADUSHIN, Education Department, Lockheed Aircraft Corporation.

Chapter XIV

FRACTIONS[1]

There are two good reasons for studying fractions in algebra. The first is that it will aid you in dealing with arithmetic fractions with which perhaps you have had difficulty. In operating with them sometimes you may not have known just what to do. By thinking more carefully about the processes used in dealing with fractions, as you must do in algebra, you will become more skillful with arithmetic fractions. The rules of procedure in arithmetic and algebra are the same. The other reason for the study of fractions in algebra is that fractions occur frequently in formulas and equations and you must learn how to work with them in order to proceed with your algebra course.

What a Fraction Is

What do you think of when you hear the word "fraction"? Probably a part of a whole — one half of a dollar or three quarters of a yard. That is what "fraction" meant originally. The word comes from the Latin *frangere* meaning "to break." But four fourths is a fraction and it is not part of a whole; it is the whole. And five fourths is more than a whole. So you see, *fraction* has come to mean something different from part of a whole.

You should think of a fraction as one number divided by another. A fraction is an indicated division. It may interest you to know that earlier peoples had so much trouble with fractions that they made every effort to avoid them. They used cumbersome compound numbers like 2 lb. 5 oz. instead of $2\frac{5}{16}$ pounds. The modern symbolism for indicating fractions has helped to make operations with them much simpler.

[1] To the Teacher. See Note 20 on page 461.

Exercises

1. What is meant by $\frac{1}{2}$ of a line?

2. Draw a line 3 inches long and divide it into 3 equal parts. What is meant by $\frac{2}{3}$ of a line?

3. If a line is divided into six equal parts, one part is _?_ of the line, two parts are _?_ of the line, and three parts are _?_ of the line. $\frac{2}{6} = \frac{?}{3}, \frac{3}{6} = \frac{?}{2}.$

4. Imagine a line 8 inches long divided into 16 equal parts. One part is _?_ of the line. $\frac{2}{16} = \frac{?}{8}, \frac{4}{16} = \frac{?}{8} = \frac{?}{4}, \frac{6}{16} = \frac{?}{8}, \frac{8}{16} = \frac{?}{8} = \frac{?}{4} = \frac{?}{2}, \frac{10}{16} = \frac{?}{8}, \frac{12}{16} = \frac{?}{8} = \frac{?}{4}, \frac{14}{16} = \frac{?}{8}.$

5. Is $\frac{1}{4}$ of a line greater or less than $\frac{1}{2}$ of the line? Is $\frac{1}{5}$ of a line greater or less than $\frac{1}{6}$ of the line?

6. If the numerators of two fractions are the same and the denominator of the first is greater than the denominator of the second, the value of the first fraction is _?_ than the value of the second fraction. Illustrate.

7. How many eighths of a line does it take to make *three* fourths of the line?

8. How many tenths does it take to make *three* fifths? How many ninths does it take to make *two* thirds?

9. How many thirds does it take to make a whole? How many fourths? How many fifths? $\frac{?}{2} = 1, \frac{?}{6} = 1, \frac{?}{8} = 1.$

10. In the preceding exercises you have been thinking of a fraction as a part of a whole. You know that a fraction also means an indicated division. From this point of view, what do the following mean? $\frac{2}{3}, \frac{3}{5}, \frac{5}{3}, \frac{3}{2}$

11. What does $\frac{205}{16}$ mean? What does $\frac{410}{32}$ mean? Change each to a mixed number. Could you tell that $\frac{205}{16} = \frac{410}{32}$ without changing each to a mixed number?

12. Compare the numerical values of the fractions $\frac{a}{b}$ and $\frac{na}{nb}$ when (*a*) $a = 2$, $b = 3$, and $n = 4$; (*b*) $a = 5$, $b = 8$, and $n = 2$.

Equivalent Fractions

The most fundamental fact about fractions, the one upon which most of the work with fractions depends, is that the form of a fraction can be changed without changing its value. You know from your experience with a ruler that $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}$ and that $\frac{3}{4} = \frac{6}{8} = \frac{12}{16}$. If you will study these equivalent fractions, you will see that in each case the change is made by multiplying or dividing the numerator and the denominator by the same number.

Thus $\frac{4}{5} = \frac{12}{15}$ because both numerator and denominator of the first fraction have been multiplied by 3 to get the second fraction. The second fraction can be changed to the first by dividing both numerator and denominator by 3.

In general, $\frac{a}{b} = \frac{na}{nb}$, where n can be any number but zero. (Why cannot n be zero?) This rule stated in words is:

> **Multiplying or dividing both the numerator and the denominator of a fraction by the same number changes the form but not the value of the fraction.**

Exercises

(1) In each exercise below tell what has been done to the numerator and the denominator of the first fraction to get the *equivalent* (equal) fraction.

(*a*) $\frac{1}{2} = \frac{2}{4}$

(*b*) $\frac{2}{5} = \frac{6}{15}$

(*c*) $\frac{3}{7} = \frac{12}{28}$

(*d*) $\frac{a}{b} = \frac{ax}{bx}$

(*e*) $\frac{x}{y} = \frac{bx}{by}$

(*f*) $\frac{12}{16} = \frac{3}{4}$

(*g*) $\frac{6}{10} = \frac{3}{5}$

(*h*) $\frac{14}{18} = \frac{7}{9}$

(*i*) $\frac{mx}{my} = \frac{x}{y}$

(*j*) $\frac{abx}{axy} = \frac{b}{y}$

(*k*) $\frac{a}{b} = \frac{axy}{bxy}$

(*l*) $\frac{x}{3} = \frac{2x}{6}$

(*m*) $\frac{3}{x} = \frac{6}{2x}$

(*n*) $\frac{3x}{6x} = \frac{1}{2}$

(2) In the literal examples on the preceding page substitute any numbers you please for the letters to help you see that the two fractions in each case have the same value.

(3) Does $\frac{2+5}{3+5}$ equal $\frac{2}{3}$? That is, does $\frac{7}{8} = \frac{2}{3}$? Which is the larger?

(4) Does $\frac{3-1}{4-1} = \frac{3}{4}$? That is, does $\frac{2}{3}$ equal $\frac{3}{4}$? Which is the larger?

Exs. (3) and (4) illustrate the fact that *adding* the same number to the numerator and the denominator of a fraction or *subtracting* the same number from the numerator and the denominator of a fraction *does change the value.*

Unless the numerator and the denominator of a fraction are the same, adding the same number to them or subtracting the same number from them changes both the form and the value of the fraction.

Stated algebraically this says:

$$\frac{a+c}{b+c} \text{ or } \frac{a-c}{b-c} \text{ does not equal } \frac{a}{b}, \text{ unless } a = b.$$

(5) Is the value of a fraction changed by squaring it?

(6) In each exercise below, tell what has been done to the first fraction to obtain the second, and tell whether or not the value has been changed.

(*a*) $\frac{3}{5}, \frac{6}{10}$

(*b*) $\frac{a}{b}, \frac{ax}{bx}$

(*c*) $\frac{3}{4}, \frac{4}{5}$

(*d*) $\frac{a}{b}, \frac{a+1}{b+1}$

(*e*) $\frac{3}{4}, \frac{2}{3}$

(*f*) $\frac{a}{b}, \frac{a-1}{b-1}$

(*g*) $\frac{2}{3}, \frac{4}{9}$

(*h*) $\frac{a}{b}, \frac{a^2}{b^2}$

(*i*) $\frac{5}{8}, \frac{2}{5}$

(*j*) $\frac{a}{b}, \frac{a-y}{b-y}$

(*k*) $\frac{a}{b}, \frac{2a}{b}$

(*l*) $\frac{a}{b}, \frac{a}{2b}$

(*m*) $\frac{ad}{bc}, \frac{a}{b}$

(*n*) $\frac{ac}{bd}, \frac{a}{b}$

(*o*) $\frac{4}{25}, \frac{2}{5}$

(*p*) $\frac{1}{2}, \frac{1}{4}$

Recognizing Multiplication and Division[1]

Since changing the form of a fraction without changing its value depends upon multiplication or division, you should be able to distinguish multiplication and division from addition and subtraction. Many errors are due to a careless confusion of these processes. For example, a student may change the fraction $\frac{2x+5}{3a+5}$ thus:

$$\frac{2x+5}{3a+5} = \frac{2x+\cancel{5}}{3a+\cancel{5}} = \frac{2x}{3a}$$

He may cross out the 5's in the belief that he is *dividing* numerator and denominator by 5. He has not divided but has *subtracted* 5 from both numerator and denominator and has therefore changed the value of the fraction.

(1) If I cross off the 5 in $3 + 5$, have I subtracted 5 from $3 + 5$ or have I divided $3 + 5$ by 5? How can I tell easily? (5 was originally *added*. If it is no longer there, I have *subtracted* it. Subtraction is the inverse of addition.)

(2) If I cross off the 5 in 3×5, have I subtracted 5 from 3×5 or have I divided 3×5 by 5? How can I tell easily? (3 was originally *multiplied* by 5. If the 5 is no longer there, I have *divided* by 5. Division is the inverse of multiplication.)

(3) In each of the following tell what has been done to the first expression to get the second:

(*a*) $7 - 3, 7$	(*g*) $2a + 3, 2a$
(*b*) $7 + 3, 7$	(*h*) $2a - 3, 2a$
(*c*) $7 \times 3, 7$	(*i*) $3(2a), 2a$
(*d*) $7 \div 3, 7$	(*j*) $(a + b) + (a - b), a + b$
(*e*) $5a, 5$	(*k*) $(a + b)(a - b), a + b$
(*f*) $5a, a$	(*l*) $(a + b) - (a - b), a + b$

You should also note that when an expression is in factored form, it is easy to see what it has been multiplied or divided by to get another expression.

(4) What has $2ab$ been multiplied by to get $4a^2bc$? You can get the answer by inspection. It is $2ac$.

[1] To the Teacher. See Note 21 on page 462.

(5) What has $(a + b)(a - b)$ been multiplied by to get $3(a + b)(a - b)^2$? Answer, $3(a - b)$.

(6) What has $x^2 + 6x + 8$ been divided by to get $x + 4$? This expression is not in factored form; hence it is not easy to give the answer immediately. Factor and get $(x + 4)(x + 2)$. You now see that the answer is $x + 2$.

Exercises

In each exercise below, what has the first number been multiplied by to get the second? You should factor all literal expressions that are not already in factored form.

1. 7, 14

2. x, ax

3. r, rs

4. y, $abxy$

5. ab^2, a^2b^2

6. x, x^3

7. $x + y$, $ab(x + y)$

8. $a - b$, $(a - b)^2$

9. $2x$, $10ax^2$

10. $5q$, $15pq^2$

11. $m + n$, $2m + 2n$

12. $2b$, $(4ab - 2b^2)$

13. $3a$, $(6a^2 - 3ab)$

14. $a + b$, $ax + bx$

15. $x - 1$, $x^2 - 2x + 1$

16. $x + 1$, $x^2 - 1$

17. $x - 1$, $3x^2 - 3$

18. $2a + 3$, $2a^3 + 5a^2 + 3a$

What has the first number been divided by to get the second?

19. my, y

20. $(a + b)^2$, $a + b$

21. aby, y

22. x^3, x

23. $3a^2x$, $3x$

24. $10ax^2$, $2x$

25. $15pq^2$, pq

26. $3m + 3n$, 3

27. $3m + 3n$, $m + n$

28. $x^2 + 2x - 15$, $x + 5$

29. $a^2b^2 + ab^3$, ab

30. $3x^2 - xy$, x

31. $\frac{5}{6} = \frac{?}{18}$ Think: The denominator of the first fraction has been multiplied by 3 to get 18. Therefore I must multiply the numerator by 3.

32. $\frac{a}{b} = \frac{?}{b(a - b)}$ Think: The denominator has been multiplied by $a - b$; hence I must multiply the numerator by $a - b$.

33. $\frac{m^3n}{2\,m^2n} = \frac{?}{2}$ Think: The denominator has been divided by m^2n; hence I must divide the numerator by m^2n.

Supply the missing terms and be prepared to state how each fraction has been changed to its equivalent fraction:

34. $\frac{4}{5} = \frac{?}{10}$

35. $\frac{7}{8} = \frac{14}{?}$

36. $\frac{mx}{my} = \frac{?}{y}$

37. $\frac{ab}{ax} = \frac{b}{?}$

38. $\frac{a}{x} = \frac{?}{mx}$

39. $\frac{rs}{2\,ts} = \frac{r}{?}$

40. $\frac{a}{5} = \frac{?}{10}$

41. $\frac{3}{n} = \frac{?}{nx}$

42. $\frac{xy}{3} = \frac{?}{6}$

43. $\frac{2\,m + n}{5} = \frac{?}{10}$

44. $\frac{y + 3}{2\,e} = \frac{2\,y + 6}{?}$

45. $\frac{ax}{a} = \frac{x}{?}$

46. $\frac{abx}{abxy} = \frac{?}{y}$

47. $\frac{m + n}{2\,x - y} = \frac{2\,m + 2\,n}{?}$

48. $\frac{a^2 - b^2}{(a - b)^2} = \frac{?}{a - b}$

49. $\frac{x^3 - x^2y}{x^3} = \frac{?}{x}$

50. $\frac{5\,a^2x}{10\,ax^2} = \frac{?}{2\,x}$

51. $\frac{6\,p^2q}{15\,pq^2} = \frac{?}{5\,q}$

52. Are $\frac{3\,ax^2}{5\,a^2x}$ and $\frac{3\,x}{5\,a}$ equivalent fractions?

53. Are $\frac{6\,a^2 - 3\,ab}{4\,ab - 2\,b^2}$ and $\frac{3\,a}{2\,b}$ equivalent fractions?

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54. $\frac{ax + ay}{bx + by} = \frac{a}{?}$

55. $\frac{x - y}{x + y} = \frac{x^2 - y^2}{?}$

56. $\frac{x^{n+2}}{x^{n+5}} = \frac{?}{x^3}$

57. $\frac{x^3 - 4\,x}{x^2 - 4\,x + 4} = \frac{x(x + 2)}{?}$

58. $\frac{3a - 3b}{a + b} = \frac{?}{a^2 + 2\,ab + b^2}$

59. $\frac{a^2 - b^2}{a^2 + 2\,ab + b^2} = \frac{?}{a + b}$

Reducing Fractions to Lowest Terms

In algebra, just as in arithmetic, it is often possible to express a fraction in simpler form by reducing it to lower terms. To do this, divide the numerator and the denominator by all factors common to both. You can divide by one factor at a time or do the reducing at one stroke by dividing by the product of all the factors. In order to simplify your work you should factor all literal expressions that are not already in factored form.

EXAMPLE 1. Reduce $\frac{3x^2y^3}{6x^2y}$ to lowest terms.

$$\frac{3x^2y^3}{6x^2y} = \frac{\cancel{3}\,\cancel{x^2}\,\overset{y^2}{\cancel{y^3}}}{\underset{2}{\cancel{6}}\,\cancel{x^2}\,\cancel{y}} = \frac{y^2}{2}.$$

The numerator and the denominator are already in factored form. You can divide both by 3, by x^2, and by y. The answer is $\frac{y^2}{2}$.

EXAMPLE 2. Reduce $\frac{mx + my}{nx + ny}$ to lowest terms. First factor the numerator and the denominator. You get $\frac{m(x + y)}{n(x + y)}$. Then divide numerator and denominator by $x + y$ and you get $\frac{m}{n}$. Thus, $\frac{mx + my}{nx + ny} = \frac{m}{n}$.

EXAMPLE 3. Reduce $\frac{x^2 - 1}{x^2 - 2x + 1}$ to lowest terms. Factored, $\frac{x^2 - 1}{x^2 - 2x + 1} = \frac{(x + 1)(x - 1)}{(x - 1)^2} = \frac{x + 1}{x - 1}$.

(1) Is this correct? $\frac{3x + \cancel{5}}{2x + \cancel{5}} = \frac{3\cancel{x}}{2\cancel{x}} = \frac{3}{2}$. Explain.

(2) Is this correct? $\frac{\cancel{a}x + b}{\cancel{a}y + c} = \frac{x + b}{y + c}$. Have the numerator and the denominator been divided by a?

Exercises

Reduce to lowest terms:

1. $\frac{16}{20}$
2. $\frac{24}{60}$
3. $\frac{mx}{my}$
4. $\frac{8x}{28}$
5. $\frac{3}{6b}$
6. $\frac{14c}{98}$

7. $\frac{3\,xy}{24\,y}$

8. $\frac{y^7}{y^3}$

9. $\frac{y^5}{y^8}$

10. $\frac{30\,s^2}{20\,s}$

11. $\frac{7\,x^2}{6\,x}$

12. $\frac{r^2xy}{r^2xt}$

13. $\frac{6\,ab}{9\,a^2x}$

14. $\frac{2\,a^3b^2}{6\,a^2b^2}$

15. $\frac{-\,x^2y^3}{x^3y^2}$

16. $\frac{3\,a^4b^3}{9\,a^3b^3}$

17. $\frac{48\,ab^2c}{60\,a^2b^2}$

18. $\frac{-\,9\,xy^2}{30\,x^2y}$

19. $\frac{-\,30\,a^2bc}{-\,18\,a^3c}$

20. $\frac{36\,a^2b^3x^2}{-\,60\,a^2bx^5}$

21. $\frac{a(x+y)}{b(x+y)}$

22. $\frac{24}{15(x+y)}$

23. $\frac{m(x-y)^2}{e(x-y)}$

24. $\frac{30\,a}{25(x-a)}$

25. $\frac{36\,x}{15(x+2)}$

26. $\frac{3 \cdot 5\,x}{2 \cdot 5\,y}$

27. $\frac{60\,x^2yz^4}{2 \cdot 3 \cdot 4\,x^2z^2}$

28. $\frac{15(a+b)}{6(2\,a+b)}$

29. $\frac{8(x-2\,y)}{18(2\,x-y)}$

30. $\frac{ax+ay}{ab+ac}$

31. $\frac{ac-bc}{ax-bx}$

32. $\frac{x^2+xy}{x^2-xy}$

33. $\frac{27\,xy}{9\,xy+9\,xz}$

34. $\frac{3\,x^2-xy}{xy+3\,x^2}$

35. $\frac{ax+by}{ax-by}$

36. $\frac{2\,ab+b^2}{a^2-b^2}$

37. $\frac{a^2b-ab^2}{a^2b^2+ab^2}$

38. $\frac{3\,a+3\,b}{a^2+2\,ab+b^2}$

39. $\frac{x^2-y^2}{x^2-2\,xy+y^2}$

40. $\frac{a^2-9}{a^2+5\,a+6}$

41. $\frac{2\,x^2+6\,x+4}{4\,x^2-12\,x-16}$

42. $\frac{2\,x^3-2\,x^2-4\,x}{3\,x^3+3\,x^2-18\,x}$

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43. $\frac{3\,y^n}{y^{n+1}}$

44. $\frac{4\,x^n(a+b)}{8\,x^{n-1}(a+b)^n}$

45. $\frac{R^2-r^2}{3\,r+3\,R}$

46. $\frac{6\,a^2+a-15}{6\,a^2-13\,a+6}$

Multiplication of Fractions

Multiplication of fractions in algebra is done in the same way as multiplication of fractions in arithmetic.

> **To multiply two fractions, multiply the numerator of the one by the numerator of the other and multiply the denominator of the one by the denominator of the other. All answers should be reduced to lowest terms if possible. If algebraic expressions are not in factored form, they should be factored to make reduction to lowest terms easy.**

Study the following examples:

(1) $\dfrac{2}{3} \times \dfrac{4}{5} = \dfrac{2 \times 4}{3 \times 5} = \dfrac{8}{15}$

(2) $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{a \times c}{b \times d} = \dfrac{ac}{bd}$

(3) $\dfrac{3}{7} \times \dfrac{14}{9} = \dfrac{\cancel{3} \times \overset{2}{\cancel{14}}}{\cancel{7} \times \underset{3}{\cancel{9}}} = \dfrac{2}{3}$

(4) $\dfrac{2\,x}{3\,y} \times \dfrac{3\,m}{4\,nx^2} = \dfrac{(\cancel{2}\cancel{x})(\cancel{3}\,m)}{(\cancel{3}\,y)(\underset{2}{\cancel{4}}\,n\underset{x}{\cancel{x^2}})} = \dfrac{m}{2\,nxy}$

(5) $\dfrac{2\,x + 3}{4} \times \dfrac{8}{3(4\,x^2 - 9)} = \dfrac{\cancel{(2\,x + 3)}\overset{2}{\cancel{8}}}{\cancel{4}(3)\cancel{(2\,x + 3)}(2\,x - 3)}$

$= \dfrac{2}{3(2\,x - 3)}$

(6) $\dfrac{a^2b - ab^2}{a^2 - b^2} \times \dfrac{a^2 + 2\,ab + b^2}{a^2b^2 + ab^3} = \dfrac{\cancel{a}\cancel{b}\cancel{(a - b)}\overset{\overset{1}{\cancel{a+b}}}{\cancel{(a + b)^2}}}{\cancel{(a - b)}\cancel{(a + b)}\underset{b}{\cancel{ab^2}}\cancel{(a + b)}}$

$= \dfrac{1}{b}$

The method of indicating division by drawing lines through factors common to both numerator and denominator of a fraction is called *cancellation.*

You should not confuse multiplying a fraction by a number with multiplying the numerator and the denominator of a fraction by a number. The first makes the fraction just that many times as big, but the second leaves the value of the fraction the same.

For example: $3 \times \frac{2}{5} = \frac{6}{5}$ but $\frac{3 \times 2}{3 \times 5} = \frac{6}{15} = \frac{2}{5}$.

If it helps you in multiplying a fraction by a whole number, you can put 1 as the denominator of the whole number.

This procedure is, however, unnecessary. Thus,

$$5 \times \frac{3}{7} = \frac{5}{1} \times \frac{3}{7} = \frac{5 \times 3}{1 \times 7} = \frac{15}{7}$$

and

$$2 \times \frac{a}{b} = \frac{2}{1} \times \frac{a}{b} = \frac{2\,a}{b}$$

Exercises

Multiply as indicated:

1. $2 \times \frac{1}{3}$
2. $3 \times \frac{2}{7}$
3. $5 \times \frac{3}{4}$
4. $-7 \times \frac{2}{3}$
5. $-5 \times -\frac{7}{8}$
6. $2 \times \frac{b}{c}$
7. $a \times \frac{a}{b}$
8. $b \times \frac{a}{b}$
9. $x^2 \times \frac{1}{x}$
10. $-p \times \frac{pq}{r}$
11. $m \times \frac{1}{m}$
12. $3 \times \frac{a+b}{a-b}$
13. $\frac{6}{13} \cdot \frac{5}{9}$
14. $\frac{7}{12} \cdot \frac{30}{14}$
15. $8 \cdot \frac{2}{5} \cdot \frac{15}{32}$
16. $-a \cdot \frac{a+b}{a-b}$
17. $\frac{a}{x} \cdot \frac{x}{y}$
18. $\frac{ab^2}{5} \cdot \frac{1}{ab}$
19. $xy \cdot \frac{x}{y}$
20. $3 \cdot \frac{x^2}{3}$
21. $a^2b \cdot \frac{a}{b^2}$
22. $\frac{3\,a^2}{5\,b} \cdot \frac{15\,b^3}{11\,ax}$
23. $\frac{9\,x}{y} \cdot \frac{y^2}{36\,x^2} \cdot \frac{27\,y}{18\,x}$
24. $\frac{xy}{a} \cdot \frac{1}{x}$
25. $2 \cdot 3 \cdot \frac{1}{3 \cdot 5}$
26. $x^2y^2 \cdot \frac{1}{xyz}$

27. $\frac{x}{y} \cdot \frac{my}{nx}$

28. $\frac{ab}{xy} \cdot \frac{x}{a^3b^2}$

29. $\frac{xy^2}{m^3n} \cdot \frac{m^2}{y^3}$

30. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{ax}{by} \cdot \frac{y^2}{x^2}$

31. $\frac{7\,xy^2}{8\,x^2y} \cdot \frac{16\,xz^2}{49\,y^2z}$

32. $3 \cdot \frac{x^2}{3}$

33. $3\,x \cdot \frac{c}{3\,x}$

34. $ab^2 \cdot \frac{c}{abd}$

35. $\frac{4 \cdot 6}{2 \cdot 3^2} \cdot \frac{27}{2^2 \cdot 3}$

36. $(x + 2) \cdot \frac{x + 3}{x + 1}$

37. $(x - 5) \cdot \frac{x + 5}{3(x - 5)}$

38. $(2\,x + 5) \cdot \frac{2\,x + 3}{2\,x - 5}$

39. $x^2y \cdot \frac{15}{2\,x + 2\,y} \cdot \frac{8}{5\,x^2y^2}$

40. $\frac{3\,a - 3\,b}{10\,ab} \cdot \frac{50\,a^2b^2}{a^2 - b^2}$

41. $\frac{a + 1}{a + 2} \cdot \frac{a + 2}{a + 1}$

42. $\frac{a + 3}{a - 3} \cdot \frac{1}{a + 3}$

43. $(n + 3) \cdot \frac{n + 3}{n + 5}$

44. $(2\,x - 5) \cdot \frac{x + 7}{2\,x - 5}$

45. $n(a + b) \cdot \frac{1}{m(a + b)}$

46. $\frac{a^2 - 4\,b^2}{a^2 - b^2} \cdot \frac{3\,a^2b^2}{a + b}$

47. $\frac{a^2 + 5\,a}{a^2 - 16} \cdot \frac{a^2 - 4\,a}{a^2 - 25}$

48. $\frac{3\,x^2 - 48\,y^2}{2\,x^2 - 8\,y^2} \cdot \frac{3\,x + 6\,y}{3\,x + 12\,y}$

49. $\frac{x^2 + 8\,x + 16}{x^2 - 9} \cdot \frac{x - 3}{x + 4}$

50. $\frac{a^2 - 3\,a - 10}{(a - 2)^2} \cdot \frac{a - 2}{a - 5}$

51. $\frac{9 - x^2}{x + 3} \cdot \frac{x}{3 - x}$

52. $\frac{24\,x^2}{3(x^2 - 4\,x + 4)} \cdot \frac{3\,x - 6}{2\,x}$

53. $\frac{x^2 - 6\,x + 5}{x - 1} \cdot \frac{x - 1}{x - 5}$

54. $\frac{c^2 - 6\,c}{c - 6} \cdot \frac{c + 3}{c}$

55. $\frac{x^2 - 24 - 2\,x}{x^2 - 30 - x} \cdot \frac{(x + 5)}{x^2 - 16}$

56. $\frac{9 - y^2}{r^3 - r} \cdot \frac{r - 1}{y + 3}$

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57. $\frac{a^2 + 7\,ab + 10\,b^2}{a^2 + 6\,ab + 5\,b^2} \cdot \frac{a + b}{a^2 + 4\,ab + 4\,b^2} \cdot \frac{a + 2\,b}{1}$

58. $\frac{x^2 - y^2}{x^2 - 3\,xy + 2\,y^2} \cdot \frac{xy - 2\,y^2}{y^2 + xy} \cdot \frac{x(x - y)}{(x - y)^2}$

59. $\frac{2\,a - 3\,b}{a^2 + 4\,ab + 4\,b^2} \cdot \frac{4\,a^2 - 4\,b^2}{4\,a^2 - 9\,b^2} \cdot \frac{5\,a^2 + 10\,ab}{3\,ab - 3\,b^2}$

Dividing by a Fraction

In algebra as in arithmetic, to divide by a fraction we multiply by its reciprocal (the inverted fraction). Thus, to divide by $\frac{1}{2}$, we multiply by 2. To divide by $\frac{a}{b}$, we multiply by $\frac{b}{a}$. Similarly, to divide a fraction by a whole number, we multiply the fraction by the reciprocal of the whole number.

EXAMPLE 1. Divide $\frac{5\,xy^2}{2\,z}$ by $\frac{3\,x^2y}{8\,tz^2}$.

SOLUTION.

$$\frac{5\,xy^2}{2\,z} \div \frac{3\,x^2y}{8\,tz^2} = \frac{5\,xy^2}{2\,z} \times \frac{8\,tz^2}{3\,x^2y} = \frac{5\,\cancel{xy^2}^{\,y}}{\cancel{2\,z}} \times \frac{\cancel{8\,tz^2}^{\,4\,z}}{3\,\cancel{x^2y}_{\,x}} = \frac{20\,tyz}{3\,x}.$$

EXAMPLE 2. Divide $\frac{5\,x - 5\,y}{xy + y^2}$ by $5\,x^2 - 5\,y^2$.

SOLUTION.

$$\frac{5\,x - 5\,y}{xy + y^2} \div (5\,x^2 - 5\,y^2) = \frac{5\,x - 5\,y}{xy + y^2} \times \frac{1}{5\,x^2 - 5\,y^2}$$

$$= \frac{\cancel{5(x - y)}}{y(x + y)} \times \frac{1}{\cancel{5}(x + y)\cancel{(x - y)}}$$

$$= \frac{1}{y(x + y)^2}$$

Exercises

Divide as indicated:

1. $\frac{x}{2} \div \frac{x}{4}$
2. $\frac{x}{4} \div \frac{x}{2}$
3. $\frac{2}{x} \div \frac{4}{x}$
4. $\frac{4}{x} \div \frac{2}{x}$
5. $\frac{x}{y} \div \frac{3}{y}$
6. $\frac{ad}{bc} \div \frac{a}{b}$
7. $\frac{a}{b} \div \frac{c}{d}$
8. $\frac{xy}{ab} \div \frac{xy}{bc}$
9. $\frac{1}{a} \div \frac{a}{b}$
10. $\frac{a}{x} \div \frac{c}{x}$
11. $\frac{x}{t} \div \frac{1}{t}$
12. $\frac{a}{x} \div \frac{x}{c}$
13. $\frac{1}{f} \div \frac{1}{F}$
14. $\frac{3}{5} \div \frac{6}{7}$
15. $\frac{8}{3\,a} \div 16$
16. $\frac{5\,x^2}{8\,xy} \div 15\,x$
17. $\frac{9\,ab^2}{8\,xy^2} \div \frac{3\,b^2}{2\,xy}$
18. $\frac{14\,a^2}{10\,b^2} \div \frac{21\,a^2}{15\,b^2}$

19. $\frac{5x}{12yz^2} \div \frac{15x^3}{18y^2z^2}$

20. $\frac{25x^2}{(3yz)^2} \div \frac{100x^3}{18yz^3}$

21. $\frac{x+3}{a} \div x^2$

22. $\frac{(a-b)(a+b)}{(x-y)(x+y)} \div \frac{a-b}{x+y}$

23. $\frac{a^2-b^2}{x^2-y^2} \div \frac{a+b}{x-y}$

24. $\frac{(2x)^3}{(4yz)^2} \div \frac{16x^2}{8y^2z^3}$

25. $\frac{2 \cdot 4}{5} \div \frac{3 \cdot 6}{15}$

26. $\frac{3(x+y)^2}{x-y} \div 6(x+y)$

27. $\frac{x-y}{x+y} \div \frac{5x^2-5y^2}{3x-3y}$

28. $\frac{x^2+2x+1}{3x} \div (x+1)$

29. $\frac{a^3-6a^2+8a}{5x} \div \frac{2a-4}{10a-40}$

30. $\frac{5a^2-5ab}{ab+b^2} \div \frac{5a^2-5b^2}{b}$

31. $\frac{4x^2-4xy-3y^2}{3x^2y} \div \frac{2x^2-3xy}{6x^3}$

32. $\frac{x^4-y^4}{x+y} \div \frac{x^2+y^2}{y}$

Changing Improper Fractions to Whole or Mixed Numbers

You will find no difficulty in changing an improper fraction to a whole or a mixed number if you remember that a fraction is an indicated division.

(1) What does $\frac{23}{5}$ mean? Change this fraction to a mixed number by dividing 23 by 5. Similarly, $\frac{x^2+3x+2}{x}$ means division. Remember the principle: To divide a polynomial by a number, divide every term by that number. The answer is $x+3+\frac{2}{x}$.

(2) What does $\frac{194}{27}$ mean? Change this fraction to a mixed number. Long division is necessary. Similarly, if the denominator of a fraction is a binomial (or other polynomial), long division is necessary. Change $\frac{x^2+5x+2}{x+3}$ to a mixed number. (Review page 213 if necessary.) The answer is $x+2-\frac{4}{x+3}$.

Exercises

Change the following fractions to whole or mixed numbers:

1. $\frac{35}{3}$
2. $\frac{40}{8}$
3. $\frac{37}{5}$
4. $\frac{45}{7}$
5. $\frac{342}{23}$
6. $\frac{782}{46}$
7. $\frac{x^2 + 3x}{x}$
8. $\frac{4a^2 - 8a + 3}{2a}$
9. $\frac{3b^3 - 7b^2 - b}{b}$
10. $\frac{2n^2 - 3n + 5}{n - 3}$
11. $\frac{8x^2 + 2x}{2x + 2}$
12. $\frac{x - 5}{x - 2}$

Combining Fractions Having the Same Denominator

The first important principle in adding (or subtracting) fractions is:

You can combine fractions by addition and subtraction only when the fractions have the same denominators.

You cannot combine fractions by addition or subtraction when they have different denominators any more than you can combine $2a$ and $3b$. The denominators can be made the same, however, and then they can be combined.

We shall first give you practice in combining fractions which have the same denominator. Becoming skillful in adding such fractions will save you much time in learning how to add fractions having different denominators.[1]

To add (or subtract) fractions having the same denominator, add (or subtract) the numerators and write the result over the denominator.

EXAMPLE. $$\frac{7a + 3b}{a - 1} + \frac{2a - 5b}{a - 1} = \frac{7a + 3b + 2a - 5b}{a - 1}$$
$$= \frac{9a - 2b}{a - 1}.$$

[1] TO THE TEACHER. See Note 22 on page 462.

Exercises

Add the following fractions:

1. $\frac{3}{7}+\frac{2}{7}$
2. $\frac{3}{a}+\frac{2}{a}$
3. $\frac{3}{a^2}+\frac{2}{a^2}$
4. $\frac{3}{a+b}+\frac{2}{a+b}$
5. $\frac{3}{a^2-b^2}+\frac{2}{a^2-b^2}$
6. $\frac{3}{a^2b}+\frac{2}{a^2b}$
7. $\frac{3}{a(a+1)}+\frac{2}{a(a+1)}$
8. $\frac{3}{(a+1)(a-1)}+\frac{2}{(a+1)(a-1)}$
9. $\frac{3}{(a-b)^2}+\frac{2}{(a-b)^2}$
10. $\frac{3}{a^2+2ab}-\frac{2}{a^2+2ab}$
11. $\frac{5a+4b}{a}+\frac{2a-3b}{a}$
12. $\frac{2a-7b}{b}+\frac{-5a+2b}{b}$
13. $\frac{7n+3}{5}+\frac{8n-7}{5}$
14. $\frac{-5n+2}{a^2b}+\frac{5n-2}{a^2b}$
15. $\frac{x^2+5}{x+1}+\frac{6-x^2}{x+1}$
16. $\frac{9a+7}{3a}+\frac{3a-7}{3a}$
17. $\frac{4n-5}{n-1}+\frac{4-3n}{n-1}$
18. $\frac{-2a+2}{2a-3}+\frac{8a-11}{2a-3}$
19. $\frac{2(a+4)}{a}+\frac{3(2a-1)}{a}$

Put Ex. 19 in the form $\frac{2a+8}{a}+\frac{6a-3}{a}$ and proceed as before.

20. $\frac{16m+n}{m-n}+\frac{5(2n-3m)}{m-n}$
21. $\frac{3(2x-5)}{x^2+1}+\frac{4(-3x+7)}{x^2+1}$
22. $\frac{x^2+3x-5}{x+2}+\frac{3(x^2-1)}{x+2}$
23. $\frac{x^2+5x}{x+1}+\frac{2(1-x)}{x+1}$
24. $\frac{2(a^2-ab)}{a+b}+\frac{2(ab-b^2)}{a+b}$
25. $\frac{5(3n+7)}{m^2n}+\frac{2(3-2n)}{m^2n}$

26. $\frac{(a+1)(a-1)}{a}+\frac{3(2a-3)}{a}$

Write Ex. 26 in the form $\frac{a^2-1}{a}+\frac{6a-9}{a}$ and proceed as before.

27. $\frac{(3x-4)^2}{2x+1}+\frac{(2x-3)(x+3)}{2x+1}$

28. $\frac{(2n+3)(2n-3)}{n}+\frac{(3n-5)(2n-7)}{n}$

29. $\frac{3a+2}{a}-\frac{a-3}{a}$

Here you are asked to subtract the second fraction from the first. You should therefore subtract the numerator $a-3$ from the numerator $3a+2$. In order to avoid errors in signs, write the exercise in the form $\frac{(3a+2)-(a-3)}{a}$ and then proceed.

30. $\frac{5a+2b}{a}-\frac{2a-3b}{a}$

31. $\frac{2a-5b}{b}-\frac{-5a-2b}{b}$

32. $\frac{-5n+2}{m^2n}-\frac{5n-2}{m^2n}$

33. $\frac{7n+1}{5}-\frac{5n+7}{5}$

34. $\frac{x^2+2}{x+1}-\frac{6-x^2}{x+1}$

35. $\frac{8a+5}{3a}-\frac{2a+5}{3a}$

36. $\frac{3a^2+5}{a+1}-\frac{2a^2+6}{a+1}$

37. $\frac{-3n+1}{2n-3}-\frac{8n-2}{2n-3}$

38. $\frac{5(a+1)}{b}-\frac{2(2a-3)}{b}$

Write first in the form $\frac{5a+5}{b}-\frac{4a-6}{b}$ and then proceed.

39. $\frac{3(2m-5n)}{m-n}-\frac{2(-m+3n)}{m-n}$

40. $\frac{a+b}{a-b}-\frac{2(3a-b)}{a-b}$

41. $\frac{2n+3}{5}+\frac{5n-2}{5}-\frac{n+6}{5}$

42. $\frac{3(2a+7)}{a+1}-\frac{-7(2a-4)}{a+1}+\frac{6a-5}{a+1}$

Be sure you know how to add fractions with the same denominator before you proceed to the next section.

Adding Fractions with Different Denominators

To add or subtract fractions with different denominators, you must change them to equivalent fractions which have the same denominator. Two or more fractions having the same denominator are said to have a *common denominator.*

In adding (or subtracting) fractions, the smallest possible common denominator should be chosen. When the denominators are arithmetic numbers, the procedure is comparatively simple.

EXAMPLE 1. Add $\frac{2a+3}{6} + \frac{5a-7}{9}$

For the common denominator choose 18, the smallest number which contains 6 and 9 as factors.

Write the next step first without numerators:

$$\frac{\qquad}{18} + \frac{\qquad}{18}$$

Then fill in the numerators, thinking as follows:

The denominator of the first fraction has been multiplied by 3; hence I must multiply the numerator by 3. Thus the numerator of the first fraction becomes $3(2a+3)$. The denominator of the second fraction has been multiplied by 2; hence I must multiply the numerator by 2. The numerator of the second fraction becomes $2(5a-7)$.

You now have two fractions with the same denominator. Combine these fractions by adding the numerators, as shown:

$$\begin{aligned} \frac{2a+3}{6} + \frac{5a-7}{9} &= \frac{3(2a+3)}{18} + \frac{2(5a-7)}{18} \\ &= \frac{6a+9}{18} + \frac{10a-14}{18} \\ &= \frac{6a+9+10a-14}{18} \\ &= \frac{16a-5}{18} \end{aligned}$$

EXAMPLE 2. $\frac{2a-3}{4} + \frac{6a+2}{3} - \frac{5a-2}{2}$

For the common denominator choose 12, the smallest number which contains 4, 3, and 2 as factors.

Write first as: $\frac{\qquad}{12} + \frac{\qquad}{12} - \frac{\qquad}{12}$

Then write in the numerators, thinking of the numbers by which you must multiply them as you did in Example 1.

$$\frac{3(2a-3)}{12}+\frac{4(6a+2)}{12}-\frac{6(5a-2)}{12}$$
$$=\frac{6a-9}{12}+\frac{24a+8}{12}-\frac{30a-12}{12}$$

When you write the next step, put parentheses around the numerators to avoid difficulties in signs.

$$=\frac{(6a-9)+(24a+8)-(30a-12)}{12}$$

Removing the parentheses,

$$=\frac{6a-9+24a+8-30a+12}{12}=\frac{11}{12}$$

When adding a whole number to a fraction, it may help you to put a denominator of 1 under the whole number. Thus,

$$3+\frac{a}{3}=\frac{3}{1}+\frac{a}{3}=\frac{3(3)}{3}+\frac{a}{3}=\frac{9+a}{3}$$

(1) What is the smallest number that will contain as factors each of the following sets of numbers?

(*a*) 3, 9
(*b*) 2, 4
(*c*) 6, 12
(*d*) 5, 15
(*e*) 2, 3
(*f*) 3, 5
(*g*) 3, 4
(*h*) 6, 4
(*i*) 6, 8
(*j*) 9, 6
(*k*) 6, 15
(*l*) 2, 4, 3

Exercises

Copy, and fill in the numerators:

1. $\frac{2}{3}=\frac{}{9}$

2. $\frac{5b}{4}=\frac{}{20}$

3. $\frac{3x}{2}=\frac{}{8}$

4. $\frac{3a+4b}{3}=\frac{}{6}$

5. $\frac{4c-5d}{10}=\frac{}{30}$

6. $5=\frac{}{4}$

7. $2a+3b=\frac{}{4}$

Combine as the signs indicate:

8. $\frac{4}{9}-\frac{2}{3}$

9. $\frac{1}{3}+\frac{1}{4}$

10. $\frac{1}{2}+\frac{3}{8}$

11. $a+\frac{3}{a}$

12. $\frac{5x}{4}+\frac{3x}{2}$

13. $\frac{15a}{8}+\frac{7a}{4}$

14. $\frac{6x}{3} - \frac{5x}{6}$

15. $\frac{xy}{5} - \frac{xz}{10}$

16. $\frac{ab}{2} - \frac{bc}{3}$

17. $\frac{x^2}{2} + \frac{y^2}{3}$

18. $\frac{3a}{4} - \frac{5b}{6}$

19. $\frac{3a}{5} + \frac{a}{4}$

20. $\frac{2}{3}x - \frac{3}{4}y$

21. $b - \frac{5a}{b}$

22. $\frac{1}{3} + \frac{3}{4} - \frac{5}{6}$

23. $\frac{2}{3} - \frac{1}{4} - \frac{1}{5}$

24. $\frac{5b}{6} + \frac{b}{9} - \frac{5b}{3}$

25. $\frac{a}{4} - \frac{b}{3} + \frac{c}{2}$

26. $\frac{n}{2} - \frac{3n}{4} + \frac{5n}{6}$

27. $\frac{3}{4}n - \frac{1}{2}n + \frac{5}{6}n$

28. $\frac{a+b}{8} + \frac{a-b}{4}$

29. $\frac{2a+b}{6} - \frac{3a-2b}{4}$

30. $\frac{3x-5}{4} + \frac{5x-3}{3}$

31. $\frac{4n-1}{5} - \frac{n+2}{4}$

32. $\frac{5x-3}{9} - \frac{3x+11}{6}$

33. $\frac{3s+5}{6} - \frac{s-3}{8}$

34. $\frac{3a-2b}{6} + \frac{a-5b}{15}$

35. $\frac{3a+1}{2} - \frac{a+3}{8} + \frac{2a-5}{4}$

36. $\frac{b+2}{2} + \frac{3b-5}{4} - \frac{b-3}{3}$

37. $\frac{4p-3}{4} - \frac{2p+7}{5} - \frac{p+2}{2}$

38. $\frac{2a-3}{5} - \frac{3a+5}{10} - \frac{a-5}{2}$

39. $\frac{x^2-y^2}{3} + \frac{x^2}{4} - \frac{x^2+y^2}{8}$

40. $\frac{3}{5} + \frac{4m+3}{10} - \frac{8m}{15}$

41. $\frac{3c}{8} - \frac{b+c}{3} + \frac{c-b}{6}$

42. $\frac{2x+3}{12} - \frac{3x-2}{6} + \frac{3}{4}$

43. $\frac{5a+1}{5} - \frac{7}{4} + \frac{2a}{3}$

44. $\frac{x^2-9}{4} - \frac{x+2}{16} + \frac{2x^2+1}{2}$

45. $2a + 3b + \frac{5}{ab}$

46. $x - 1 + \frac{7}{2x^2y}$

Adding Fractions with Literal Monomial Denominators

EXAMPLE 1. Combine: $\frac{a}{b} + \frac{c}{d}$

Here the lowest common denominator is *bd*, the product of the denominators. Whenever the denominators have no common factor, the lowest common denominator is the product of the denominators.

Then $$\frac{a}{b} + \frac{c}{d} = \frac{?}{bd} + \frac{?}{bd}$$

The first denominator has been multiplied by *d*; hence you must multiply the numerator by *d*. This gives *ad* as the new numerator of the first fraction. The second denominator has been multiplied by *b*; hence you must multiply the numerator by *b*. This gives *bc* as the new numerator of the second fraction.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

EXAMPLE 2. Combine: $\frac{3}{x^2} + \frac{5}{2\,xy} - \frac{4}{3\,y^2}$

The lowest common denominator must be evenly divisible by each of the denominators. 6 will contain 2 and 3; x^2 will contain x^2 and the x in $2\,xy$; y^2 will contain the y^2 in $3\,y^2$ and the y in $2\,xy$. Hence $6\,x^2y^2$ is the lowest common denominator.

Then, $$\frac{3}{x^2} + \frac{5}{2\,xy} - \frac{4}{3\,y^2} = \frac{18\,y^2}{6\,x^2y^2} + \frac{15\,xy}{6\,x^2y^2} - \frac{8\,x^2}{6\,x^2y^2}$$
$$= \frac{18\,y^2 + 15\,xy - 8\,x^2}{6\,x^2y^2}$$

Exercises

If the following were denominators, what would you use in each case for the lowest common denominator?

1. a, b
2. x^2, x
3. x, y, z
4. x^2, xy, y^2
5. $2\,a, 3\,a$
6. $6\,x, 3\,x$
7. $6\,p, 4\,p$
8. $2\,x, 3\,y$
9. ab^2, a^2b

Combine as the signs indicate:

10. $\frac{1}{a}+\frac{1}{b}$

11. $\frac{2}{a}+\frac{3}{b}$

12. $\frac{1}{a}-\frac{1}{b}$

13. $\frac{4}{x}-\frac{5}{y}$

14. $\frac{a}{b}-\frac{c}{d}$

15. $\frac{3}{2c}+\frac{4}{6c}$

16. $\frac{5}{4a}-\frac{3}{8a}$

17. $3+\frac{1}{a}$

18. $ax+\frac{b}{x}$

19. $3+\frac{a}{b}$

20. $2x-\frac{x+y}{y}$

21. $\frac{7a}{10p}-\frac{2b}{5p}$

22. $\frac{5r}{4c}+\frac{4s}{5d}$

23. $\frac{3}{x}+\frac{5}{x^2}$

24. $\frac{4a}{b^2}-\frac{3a}{b}$

25. $\frac{3}{x}-\frac{5}{x^3}+\frac{2}{x^2}$

26. $\frac{a}{x^3}+\frac{b}{x^2}-\frac{c}{x}$

27. $\frac{5}{ab^2}-\frac{7}{a^2b}$

28. $a-\frac{b}{c}$

29. $\frac{a}{x}-(a-1)$

30. $a^2-\frac{1}{a}$

31. $\frac{2}{xy}+\frac{3}{yz}$

32. $\frac{9}{mn}+\frac{3}{mn^2}$

33. $\frac{8}{x}+\frac{3}{xy}$

34. $\frac{3}{x^2}-\frac{5}{xy}+\frac{2}{y^2}$

35. $\frac{x^2}{y^2}-\frac{y^2}{x^2}$

36. $\frac{5}{a^2}+\frac{7}{ab}+\frac{1}{b^2}$

37. $\frac{1}{6p}-\frac{1}{4p}+\frac{1}{3p}$

38. $\frac{3}{b}+\frac{5}{2b}-\frac{11}{3b}$

39. $\frac{a+b}{b}-\frac{a-b}{a}$

40. $\frac{x+1}{2x}+\frac{2}{x}$

41. $\frac{2x-1}{x}-\frac{x+3}{3x}$

42. $v+\frac{m_1v_1}{m}$

43. $\frac{1}{2a^2}-\frac{5}{6ab}+\frac{7}{12b^2}$

44. $\frac{x}{2b}+\frac{2x}{b}-\frac{1}{b^2}$

45. $\frac{x-1}{3xy}+\frac{x^2-x}{9x^2y^2}$

46. $\frac{2a+1}{a}+\frac{3b+5}{b}$

47. $\frac{3a+5}{a}-\frac{2b-3}{b}$

48. $\frac{3x-y}{x^2}-\frac{4x+2y}{x}$

49. $\frac{4x+y}{x}-\frac{3x-4y}{x^2}$

50. $\frac{2y-3}{2y}+\frac{y-4}{3y}$

51. $\frac{2}{x^2}-\frac{1-x}{xy}+\frac{y-2}{y^2}$

52. $\frac{3}{a^2}-\frac{1-b}{ab^2}+\frac{1}{b^2}$

Adding Fractions with Binomial Denominators

EXAMPLE 1. Combine: $\dfrac{4x}{x-3} - \dfrac{2}{x+3}$

Since these denominators have no common factor, the lowest common denominator is the product $(x-3)(x+3)$. The first denominator is multiplied by $x+3$ to get the new common denominator. Therefore, the numerator of the first fraction must be multiplied by $x+3$. This gives $4x(x+3)$. The denominator of the second fraction is multiplied by $x-3$ to get the new common denominator; consequently its numerator must be multiplied by $x-3$. This gives $2(x-3)$.

Then
$$\frac{4x}{x-3} - \frac{2}{x+3} = \frac{4x(x+3)}{(x-3)(x+3)} - \frac{2(x-3)}{(x-3)(x+3)}$$
$$= \frac{(4x^2+12x) - (2x-6)}{(x-3)(x+3)}$$
$$= \frac{4x^2+12x-2x+6}{(x-3)(x+3)}$$
$$= \frac{4x^2+10x+6}{(x-3)(x+3)}$$

EXAMPLE 2. Combine: $\dfrac{5}{3a-6} - \dfrac{3}{4a-8}$

Always factor the denominators when it is possible.

Then
$$\frac{5}{3a-6} - \frac{3}{4a-8} = \frac{5}{3(a-2)} - \frac{3}{4(a-2)}$$

$12(a-2)$ contains the factors 3, 4, and $a-2$. Change to equivalent fractions with this denominator and combine.

$$\frac{5}{3(a-2)} - \frac{3}{4(a-2)} = \frac{20}{12(a-2)} - \frac{9}{12(a-2)}$$
$$= \frac{11}{12(a-2)}$$

It is simple to choose the lowest common denominator when the denominators are in factored form.

EXAMPLE 3. If $a^2b + ab^2, a^2 - b^2$, and $(a - b)^2$ were denominators, what would be the lowest common denominator? Factor the expressions and get $ab(a + b)$, $(a - b)(a + b)$, and $(a - b)^2$. Now by inspection you see that you need the factors a, b, $a + b$, and $a - b$. But in order to take care of the last expression, you need $a - b$ to the second power. The lowest common denominator is therefore $ab(a + b)(a - b)^2$.

If the following were denominators, what would you use in each case for the lowest common denominator?

(a) $x + 5, x - 5$
(b) $a + 4, a + 2$
(c) $2x + 2y, x + y$
(d) $3a + 6, 4a + 8$
(e) $2a - 4, 5a - 10$
(f) $x - 3, x^2 - 9$
(g) $2x - 6, x^2 - 9$
(h) $x^2 - 4, (x + 2)^2$

EXAMPLE 4. $\dfrac{x+1}{x^2-9} - \dfrac{4}{x+3}$

$$= \frac{x+1}{(x-3)(x+3)} - \frac{4}{x+3}$$
$$= \frac{x+1}{(x-3)(x+3)} - \frac{4(x-3)}{(x-3)(x+3)}$$
$$= \frac{x+1}{(x-3)(x+3)} - \frac{4x-12}{(x-3)(x+3)}$$
$$= \frac{(x+1)-(4x-12)}{(x-3)(x+3)}$$
$$= \frac{x+1-4x+12}{(x-3)(x+3)}$$
$$= \frac{-3x+13}{(x-3)(x+3)}$$

Exercises

Combine as the signs indicate:

1. $\dfrac{5}{x+2} + \dfrac{3}{x-2}$
2. $\dfrac{5}{x+5} - \dfrac{3}{x-5}$
3. $\dfrac{2}{a+3} + \dfrac{5}{a+5}$
4. $\dfrac{2x}{x-y} - \dfrac{3y}{x+y}$
5. $\dfrac{a}{2a+2b} - \dfrac{b}{3a+3b}$
6. $\dfrac{4a}{6a-2b} + \dfrac{3b}{9a-3b}$
7. $\dfrac{2}{3r+3s} + \dfrac{3}{5r-5s}$
8. $\dfrac{3x}{2y-3} - \dfrac{2x}{3y-2}$

9. $\frac{x+3}{x-5}+\frac{x-5}{x+3}$

10. $\frac{a-2}{a+3}-\frac{a-3}{a+5}$

11. $\frac{x+3}{x^2-4}+\frac{x-5}{x+2}$

12. $\frac{3a+2}{3a+6}-\frac{a-2}{a^2-4}$

13. $\frac{a+b}{ax+ay}-\frac{a+b}{bx+by}$

14. $\frac{a-b}{a+b}-\frac{a}{a^2-b^2}$

15. $\frac{x-y}{x+y}+\frac{4xy}{x^2-y^2}$

16. $\frac{x+y}{x-y}-\frac{4xy}{x^2-y^2}$

17. $\frac{5}{3x-3}+\frac{x}{2x+2}-\frac{3x^2}{x^2-1}$

18. $\frac{x^2}{x+y}-\frac{y^2}{x-y}-(x-y)$

Signs in a Fraction*

The sign of a fraction is the plus or minus sign before the fraction. In $+\frac{-2}{+3}$, the sign of the fraction is +. In $-\frac{-2}{+3}$, the sign of the fraction is −. A fraction, therefore, involves three signs: (1) the sign of the numerator, (2) the sign of the denominator, and (3) the sign of the fraction. *You can change any two of these signs without changing the value of the fraction.*

Thus $+\frac{+12}{+4}=+\frac{-12}{-4}=-\frac{+12}{-4}=-\frac{-12}{+4}.$

In each case we have changed two of the signs and yet each fraction equals + 3.

You should realize that "changing signs" means "multiplying by − 1." Hence if you are changing signs in either a numerator or a denominator which has more than one term, you must watch that you change the signs of all the terms.

Exercises

Study and explain the following examples:

1. $\frac{a+b-c}{4}=\frac{c-a-b}{-4}=-\frac{c-a-b}{4}$

2. $-\frac{a+b}{b-a}=\frac{a+b}{a-b}=-\frac{-a-b}{a-b}$

3. $\frac{3}{b+a}=\frac{3}{a+b}$

Show that the fractions in each of the following exercises are equal:

4. $\frac{-9}{x-3} = -\frac{9}{x-3}$

5. $\frac{2}{2-x} = -\frac{2}{x-2}$

6. $\frac{a}{a+b} = \frac{a}{b+a}$

7. $\frac{5}{6-x} = \frac{-5}{x-6}$

8. $\frac{7}{(x-2)(3-x)} = -\frac{7}{(x-2)(x-3)}$

9. $\frac{2}{(b-a)(d-c)} = \frac{2}{(a-b)(c-d)}$

10. $\frac{a+b}{d-c} = \frac{-a-b}{c-d}$

11. $\frac{(2-x)^2}{3} = \frac{(x-2)^2}{3}$

Exercises

Combine these fractions:

12. $\frac{a}{a-b} - \frac{b}{b-a}$ $\left(\text{Write first as } \frac{a}{a-b} + \frac{b}{a-b}.\right)$

13. $\frac{4}{x-2} + \frac{2}{2-x}$

14. $\frac{y}{y-3} + \frac{1}{3-y}$

15. $\frac{10}{2y-1} - \frac{6}{1-2y}$

16. $\frac{3a}{x-y} + \frac{2b}{y-x}$

17. $\frac{a^2}{a^2-1} + \frac{a}{1-a}$

18. $\frac{4}{x-3} - \frac{x}{9-x^2}$

19. $\frac{a-b}{a+b} - \frac{a-b}{b+a}$

20. $\frac{2a+3b}{2} - \frac{a+b}{-4}$

Reduce to lowest terms:

21. $\frac{a-b}{b-a}$

22. $\frac{a+b}{b+a}$

23. $\frac{a^2-b^2}{b-a}$

24. $\frac{a^2-b^2}{b+a}$

25. $\frac{a-2}{8-4a}$

26. $\frac{a+2}{8+4a}$

27. $\frac{(x-2)(x-3)}{(3-x)(2-x)}$

28. $\frac{(a-b)^2}{(b-a)^2}$

Fractions Occurring in Formulas

Below are some expressions containing fractions. Some of them occur in science formulas, particularly in the formulas of physics. You should be able to combine these fractions, as indicated.

In Exs. 2 and 3, the numbers 1, 2, and 3 are not exponents, as you might think after a hasty glance. They are called *subscripts*. r_1 is read "r sub one," r_2 is read "r sub two," and r_3 is read "r sub three." r_1, r_2, and r_3 should be thought of as three distinct symbols, just as a, b, and c are distinct symbols. Likewise d and D, or m and M, are distinct symbols.

Exercises

Combine the fractions as indicated:

1. $\dfrac{d}{V} + \dfrac{D}{V}$
2. $\dfrac{1}{r_1} + \dfrac{1}{r_2}$
3. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3}$
4. $P + \dfrac{Prt}{100}$
5. $\dfrac{1}{a} + t$
6. $a - \dfrac{am}{M + m}$
7. $ka + \dfrac{ka^2}{b - a}$
8. $k + \dfrac{kt}{273}$
9. $\dfrac{l}{Mt} - \dfrac{l}{t}$
10. $\dfrac{ad^2}{t} + d$
11. $R + \dfrac{r}{n}$
12. $\dfrac{1}{F} - \dfrac{1}{f}$

13. Find the value of $\frac{y_2 - y_1}{x_2 - x_1}$ when $y_2 = 5$, $y_1 = 3$, $x_2 = 8$, $x_1 = 4$; when $y_2 = 3$, $y_1 = -1$, $x_2 = -7$, and $x_1 = 5$.

14. Show that $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$.

15. Factor $\pi r_2^2 - \pi r_1^2$, and find its value when $r_2 = 5$ and $r_1 = 2$. (Use $\pi = \frac{22}{7}$.)

16. Find the value of $a_0x^2 + a_1x + a_2$ when $a_0 = 2$, $a_1 = -3$, $a_2 = 4$, and $x = 2$.

More Practice in Combining Fractions

1. $2 + \frac{1}{4}$
2. $3 + \frac{2}{5}$
3. $1 + \frac{4}{5}$
4. $1 + \frac{5}{y}$
5. $1 + \frac{6}{2y}$
6. $x + \frac{3}{8}$
7. $1 + \frac{x}{y}$
8. $1 - \frac{x}{y}$
9. $1 - \frac{a^2}{b^2}$
10. $1 + \frac{a^2}{b^2}$
11. $x + \frac{y}{z}$
12. $a + \frac{1}{a}$
13. $x^2 - \frac{1}{x}$
14. $z - \frac{x}{y}$
15. $a - \frac{1}{a}$
16. $\frac{(a+b)^2}{4ab} - 1$
17. $\frac{3}{x+2} + \frac{5}{x-4}$
18. $\frac{2}{y+4} + \frac{3}{y+3}$
19. $\frac{6}{y+2} + \frac{3}{y+3}$
20. $\frac{5}{b-4} + \frac{3}{b+2}$
21. $\frac{2x+1}{4} - \frac{3x-4}{8}$
22. $\frac{x-y}{z} - \frac{z-x}{y}$
23. $\frac{3}{y-2} - \frac{2}{y+5}$
24. $\frac{y+2}{4} - \frac{y-5}{5}$
25. $\frac{6}{y+4} - \frac{4}{y+8}$
26. $\frac{x+2}{x+4} - \frac{x-1}{x+6}$
27. $\frac{5}{y^2-9} + \frac{4}{y+3}$
28. $\frac{3}{y^2-4} - \frac{2}{y-2}$
29. $\frac{2a+b}{3a+3b} - \frac{a-b}{6a+6b}$
30. $\frac{a+b}{a-b} - \frac{a-b}{a+b}$
31. $\frac{a-b}{a+b} - \frac{a^2+b^2}{a^2-b^2}$
32. $\frac{3}{x^3-xy^2} + \frac{4}{x^2-xy}$
33. $\frac{6}{y^2+5y+6} - \frac{y}{y+3}$
34. $\frac{y+6}{y^2+8y+15} - \frac{y+3}{y+3}$
35. $\frac{a+2}{a-2} + \frac{3a}{(a+2)(a^2-4)}$

Complex Fractions

The division of a fraction by a fraction may be indicated thus: $\dfrac{\frac{a}{b}}{\frac{c}{d}}$. This, of course, is the same as the expression $\frac{a}{b} \div \frac{c}{d}$. Any fraction that has a fraction in its numerator or in its denominator or in both is called a *complex fraction.*

(1) Simplify $\dfrac{3 - \frac{2}{3}}{1 - \frac{1}{3}}$.

First method: $3 - \dfrac{2}{3} = \dfrac{7}{3}$; $1 - \dfrac{1}{3} = \dfrac{2}{3}$.

Hence $\dfrac{3 - \frac{2}{3}}{1 - \frac{1}{3}} = \dfrac{\frac{7}{3}}{\frac{2}{3}} = \dfrac{7}{3} \times \dfrac{3}{2} = \dfrac{7}{2}$.

Second method: Multiply both numerator and denominator of the complex fraction by 3. Then $\dfrac{3 - \frac{2}{3}}{1 - \frac{1}{3}} = \dfrac{9 - 2}{3 - 1} = \dfrac{7}{2}$.

Remember in this method that you must multiply both numerator and denominator by the same number.

For example: Simplify $\dfrac{\frac{3}{4}}{1\frac{1}{2}}$.

$\dfrac{\frac{3}{4}}{1\frac{1}{2}} = \dfrac{\frac{3}{4}}{\frac{3}{2}}$. Now multiply both numerator and denominator by 4.

$$\frac{4(\frac{3}{4})}{4(\frac{3}{2})} = \frac{3}{6} = \frac{1}{2}.$$

You may find it easier to use the first method in this type of example; that is: $\dfrac{\frac{3}{4}}{1\frac{1}{2}} = \dfrac{\frac{3}{4}}{\frac{3}{2}} = \dfrac{\cancel{3}}{4} \times \dfrac{2}{\cancel{3}} = \dfrac{2}{4} = \dfrac{1}{2}$.

Which of the two methods above do you prefer?

Exercises

Perform the divisions indicated by these complex fractions, using both the methods described above:

1. $\dfrac{\frac{1}{2}}{4}$
2. $\dfrac{\frac{1}{3}}{8}$
3. $\dfrac{20}{\frac{1}{4}}$
4. $\dfrac{\frac{a}{4}}{2}$
5. $\dfrac{\frac{2}{3}}{1\frac{1}{2}}$
6. $\dfrac{\frac{1}{2} + \frac{1}{4}}{2\frac{1}{2}}$
7. $\dfrac{\frac{1}{2} - \frac{1}{4}}{5}$
8. $\dfrac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{2} + \frac{1}{3}}$
9. $\dfrac{\frac{2}{5} + \frac{1}{4}}{10}$
10. $\dfrac{\frac{2}{3} + \frac{1}{4}}{1 + \frac{1}{2}}$
11. $\dfrac{2\frac{1}{2} - \frac{1}{4}}{\frac{1}{2} - \frac{1}{8}}$
12. $\dfrac{3 - \frac{2}{3}}{1 - \frac{1}{3}}$

13. $\dfrac{\frac{a}{b}+\frac{c}{d}}{ac}$

14. $\dfrac{\frac{a}{b}-\frac{c}{d}}{ad}$

15. $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{xy}}$

16. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{ab}}$

17. $\dfrac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b}}$

18. $\dfrac{\frac{x}{y}-\frac{y}{x}}{x+y}$

19. $\dfrac{x+y}{\frac{1}{x}+\frac{1}{y}}$

20. $\dfrac{\frac{1}{x}-\frac{1}{y}}{x^2-y^2}$

21. $\dfrac{\frac{1}{x}-1}{x-x^2}$

22. $\dfrac{\frac{1}{y}-1}{\frac{1}{y}+1}$

23. $\dfrac{\frac{a}{b}-1}{a^2-b^2}$

24. $\dfrac{1-\frac{x^2}{y^2}}{x+y}$

25. $\dfrac{x+\frac{1}{x}}{x}$

26. $\dfrac{x}{x+\frac{1}{x}}$

Chapter Summary

In this chapter you have learned how to change the form of an algebraic fraction to an equivalent fraction. You have also learned how to add, subtract, multiply, and divide fractions. A fraction is an indicated quotient — that is, one number divided by another — and the most important principle in connection with fractions is that they can be changed in form without changing their value by multiplying or dividing both numerator and denominator by the same number.

You should keep clearly in mind that fractions cannot be combined by addition or subtraction unless they have a common denominator, but in all cases they can be changed so that they will have a common denominator. In choosing a common denominator it is important to have all denominators in factored form. When they are, the common denominator can be found by inspection. In multiplication and division of fractions a common denominator is not necessary.

You should understand the following technical terms:

Fraction	Improper fraction
Reducing to lowest terms	Complex fraction
Equivalent fractions	Common denominator

Chapter Review

1. A fraction is an indicated ___?___.

2. If in a fraction the numerator increases while the denominator remains constant, the value of the fraction ___?___.

3. If in a fraction the denominator increases while the numerator remains constant, the value of the fraction ___?___.

4. To change a fraction to an equivalent fraction, I ___?___ or ___?___ both ___?___ and ___?___ by the same number.

5. Does $\frac{x+2}{y+2} = \frac{x}{y}$? Explain.

6. Does $\frac{2x}{2y} = \frac{x}{y}$? Explain.

In the exercises below, what has been done to the first number in each case to get the second?

7. $3x + 2, 3x$

8. $3x - 2, 3x$

9. $6x^2, 2x$

10. $6x + 9, 2x + 3$

11. $(a + b)(a - b), a - b$

12. $a + b, 3a(a + b)$

13. Which is the larger, $\frac{x}{y}$ or $\frac{x+1}{y}$, if both x and y are positive?

14. What do you do to reduce a fraction to lowest terms?

15. What would you do to change $\frac{3a+2}{a+1}$ to an equivalent fraction with the denominator $3(a+1)(a-1)$?

16. Is this correct? $\frac{\cancel{x}(\cancel{a}+\cancel{b})}{\cancel{x}y(\cancel{a}-\cancel{b})} = -\frac{1}{y}$. Explain.

17. Explain why $\frac{x}{y}, \frac{3x}{3y}, \frac{xy^2}{y^3}$ all have the same value. If $x = 2$ and $y = 3$, what is the value of each fraction?

18. Which is larger $\frac{x}{y}$ or $\frac{x}{y+1}$ if both x and y are positive?

Reduce to lowest terms:

19. $\frac{15a^2b}{12ab}$

20. $\frac{24ab}{6(a+b)}$

21. $\frac{18x}{6x^2(x-2)}$

22. $\frac{5a^2}{15ax-25ay}$

23. $\frac{a^2b+ab^2}{a^3b-ab^3}$

24. $\frac{5x+5}{x^2+2x+1}$

25. $\frac{4a^2-4}{4a-4}$

26. $\frac{p^2-3p-4}{2p^2-14p+24}$

*27. $\frac{a-b}{b-a}$

*28. $\frac{a-2}{6-3a}$

*29. $\frac{a^2-4}{2-a}$

30. How do you multiply one fraction by another?

31. How do you divide by a fraction?

Multiply as indicated:

32. $\frac{5}{9b}\cdot\frac{81b^2}{25}$

33. $\frac{27ab}{2xy}\cdot\frac{9x^2y^2}{ab^2}$

34. $2\cdot\frac{rs^2t}{abc}$

35. $\frac{4a-8b}{2}\cdot\frac{2a+4b}{a^2-4b^2}$

Divide as indicated:

36. $\frac{4x^2y}{3a^2b}\div\frac{8x^2yz}{7abc}$

37. $\frac{(x-y)^2}{27xy}\div\frac{x^2-y^2}{xy^2}$

38. $\frac{a^2-4}{a+3}\div(a+2)$

39. $\frac{a^4-b^4}{3ab}\div\frac{x+y}{a^2}$

Change to mixed numbers:

40. $\frac{x^2+5x-2}{x}$

41. $\frac{6x^2+2x+5}{x+2}$

42. If two fractions have the same value but not the same form, they are called _?_ fractions.

43. Does $\frac{2}{3}+\frac{3}{4}=\frac{5}{7}$? Explain.

44. How do you add or subtract two fractions which have the same denominator?

45. What must you do before you can add or subtract two fractions which do not have the same denominator?

Add or subtract as indicated:

46. $\frac{a}{3} - \frac{a}{2} + \frac{a}{4}$

47. $\frac{x}{3\,y} + \frac{3\,x}{y} - \frac{2}{y^2}$

48. $\frac{2\,a + 3\,b}{4} - \frac{a + b}{8}$

49. $a + b - \frac{3\,a}{4}$

50. $\frac{a + b}{b} - \frac{a + b}{c}$

51. $a + 2 + \frac{4}{a - 2}$

52. $\frac{1}{a} + \frac{2}{a + b} - \frac{3}{b}$

53. $\frac{5}{x^2 - 9} - \frac{x + 5}{x + 3}$

Which of the following pairs of fractions are equivalent?

*54. $\frac{a}{b}, \frac{-a}{-b}$

*55. $\frac{a}{b}, -\frac{-a}{b}$

*56. $\frac{a}{b}, -\frac{a}{-b}$

*57. $\frac{a + b}{c + d}, \frac{b + a}{d + c}$

*58. $\frac{a - b}{c - d}, \frac{b - a}{d - c}$

*59. $\frac{a}{2 - a}, -\frac{a}{a - 2}$

Perform the divisions indicated:

60. $\frac{\frac{3}{8}}{2}$

61. $\frac{\frac{3}{4}}{\frac{5}{8}}$

62. $\frac{1 - \frac{1}{2}}{2 + \frac{1}{2}}$

63. $\frac{\frac{3}{2} + \frac{1}{3}}{\frac{3}{2} - \frac{1}{3}}$

64. $\frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a} + \frac{1}{b}}$

65. $\frac{1 - \frac{a^2}{b}}{\frac{ab + a^2}{b^3}}$

Add or subtract as indicated:

66. $\frac{1}{f} + \frac{1}{F}$

67. $P + \frac{5\,P}{100}$

*68. $\frac{x}{x + 3} - \frac{x}{3 - x}$

69. $\frac{a - b}{a + b} - \frac{a + b}{a - b}$

Maintaining Skills

(*Percentage*)

Write the following as per cents:

1. $\frac{1}{2}$ 4. $\frac{1}{3}$ 7. $\frac{3}{8}$ 10. $\frac{1}{5}$
2. $\frac{1}{4}$ 5. $\frac{2}{3}$ 8. $\frac{5}{8}$ 11. $\frac{2}{5}$
3. $\frac{3}{4}$ 6. $\frac{1}{8}$ 9. $\frac{7}{8}$ 12. $\frac{5}{5}$

Write the following as per cents:

13. .50 16. .62 19. .84 22. .625
14. .25 17. .7 20. .125 23. 1.62
15. .75 18. .08 21. .375 24. 2.00

Write the following as decimals:

25. 43% 27. 135% 29. 8.7% 31. 275%
26. 2% 28. 6.5% 30. .2% 32. 200%

Compute the following. (This is Case I of percentage. The other two cases are done more easily by algebra than by arithmetic, as you have seen on pages 43 and 44.)

33. 25% of 62 39. 4.6% of 840
34. 4% of 95 40. 125% of 56
35. 7% of 1400 41. 106% of 92
36. $12\frac{1}{2}$% of 35 42. $66\frac{2}{3}$% of 42
37. 62% of 934.2 43. .5% of 300
38. $62\frac{1}{2}$% of 456 44. .3% of 500

CHAPTER XV

EQUATIONS CONTAINING FRACTIONS

The equations we meet in algebra often contain fractions. Formulas are equations, and in constructing and using them we must in many cases deal with fractions. Likewise the equations we form in the solution of different types of practical verbal problems often involve fractions. For these reasons fractional equations is an important subject in algebra, and knowing how to solve these equations is a matter of practical concern. In regard to fractional equations there is just one important new fact for you to keep in mind.

> **Fractional equations can be changed so that they contain no fractions.**

Then they may be solved like other equations. The process of getting rid of fractions in equations is called *clearing of fractions.*

Try to solve the following equation without reading further.

$$\frac{2n}{3} + \frac{3n}{4} = \frac{17}{2}$$

Method Used in Clearing of Fractions

You remember that both sides of an equation may be multiplied by the same number without destroying the equality. This is the principle which we shall now use.

(1) Solve and check: $\frac{n}{2} + \frac{n}{3} = \frac{5}{6}$. First, multiply both sides of the equation by a number that will get rid of all fractions. This number is found just as you find the lowest common denominator in adding fractions. In this case it is obviously 6, the smallest number that contains 2, 3, and 6. Think:

$$6\left(\frac{n}{2}\right) = \overset{3}{\cancel{6}} \times \frac{n}{\cancel{2}} = 3\,n;\; 6\left(\frac{n}{3}\right) = \overset{2}{\cancel{6}} \times \frac{n}{\cancel{3}} = 2\,n;\; \cancel{6} \times \frac{5}{\cancel{6}} = 5,$$

and write:

$$3\,n + 2\,n = 5$$
$$5\,n = 5$$
$$n = 1$$

CHECK. $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}.$

(2) Solve and check: $\dfrac{3\,y}{4} - \dfrac{4\,y - 5}{5} = \dfrac{7\,y - 5}{10}.$

Multiply both sides of the equation by 20.

Think:

$$20\left(\frac{3\,y}{4}\right) = \overset{5}{\cancel{20}} \times \frac{3\,y}{\cancel{4}} = 15\,y;\; 20\left(\frac{4\,y-5}{5}\right) = \overset{4}{\cancel{20}} \times \frac{4\,y-5}{\cancel{5}}$$
$$= 4(4\,y - 5);\; 20\left(\frac{7\,y-5}{10}\right) = \overset{2}{\cancel{20}} \times \frac{7\,y-5}{\cancel{10}}$$
$$= 2(7\,y - 5)$$

and write:

$$15\,y - 4(4\,y - 5) = 2(7\,y - 5)$$
$$15\,y - 16\,y + 20 = 14\,y - 10$$
$$-15\,y = -30$$
$$y = 2$$

CHECK. $\dfrac{3\,y}{4} - \dfrac{4\,y-5}{5} = \dfrac{6}{4} - \dfrac{3}{5} = \dfrac{3}{2} - \dfrac{3}{5} = \dfrac{15-6}{10} = \dfrac{9}{10}.$

$\dfrac{7\,y-5}{10} = \dfrac{14-5}{10} = \dfrac{9}{10}.$

Remember: *In solving fractional equations there should be no fractions in the second step.*

Distinguish between clearing of fractions in equations and adding fractions. When you add fractions you should get a fraction for an answer unless the sum reduces to a whole number.

(3) Solve and check: $\dfrac{2\,n}{3} + \dfrac{5\,n}{12} - \dfrac{5\,n}{9} = \dfrac{13}{2} + \dfrac{n}{6}.$

(4) Solve and check:

$$\frac{8\,x - 5}{3} + 7 = \frac{3\,x + 6}{4} - \frac{3\,x - 7}{6} + \frac{1}{4}.$$

Exercises

1. Multiply by 6: $\frac{x}{2}$, $\frac{x}{3}$, $\frac{2x+1}{6}$

2. Multiply by 12: $\frac{n}{3}$, $\frac{3n}{4}$, $\frac{5n+2}{6}$

3. Multiply by 10: $\frac{a+3}{5}$, $\frac{3a-2}{2}$

4. Multiply by 8: $\frac{1}{2}(2n+5)$, $\frac{3}{4}(n-7)$, $\frac{5}{8}(2n+1)$

5. Multiply by 6: $\frac{2x}{3} - \frac{3x}{2}$

6. Multiply by 12: $\frac{5n}{3} - \frac{3n}{4} + \frac{5n+2}{6}$

7. Multiply by 8: $\frac{3x-1}{4} - \frac{2x+5}{2}$

In Exs. 8–13 what is the smallest number by which you can multiply both sides of the equation to clear of fractions?

8. $\frac{n}{3} + \frac{n}{4} = 14$

9. $\frac{3n}{4} - \frac{5n}{2} = -7$

10. $\frac{5x}{6} + \frac{5x}{9} = \frac{1}{2}$

11. $\frac{a+2}{3} + \frac{a+3}{2} = 3$

12. $\frac{a+5}{2} - \frac{a+1}{4} = 3$

13. $\frac{4a+1}{2} - \frac{2a+3}{3} = \frac{5a-4}{4}$

14. In Exs. 8–13 multiply each side of the equation by a number that will give you a new equation without fractions.

15. Solve and check the equations in Exs. 8–13.

Solve the following equations:

16. $\frac{n}{3} = 7$

17. $\frac{2x}{5} = 3$

18. $\frac{5x}{2} = -1$

19. $\frac{n}{2} - 5 = 0$

20. $\frac{n-1}{3} = 2$

21. $\frac{2a+3}{2} = -5$

22. $\frac{3n-5}{n} = 2$

23. $\frac{x+3}{2x} = 5$

24. $\frac{2n}{3} + \frac{3n}{2} = \frac{13}{3}$

25. $\frac{3a}{5} - \frac{5a}{2} = \frac{19}{5}$

26. $\frac{3x}{4} - 2 = \frac{x}{4}$

27. $\frac{n}{2} - \frac{2n}{3} + \frac{3n}{4} = 7$

28. $3a - \frac{4a}{5} = 22$

29. $\frac{x+1}{2} = \frac{x-5}{3}$

30. $\frac{6b-7}{5} = \frac{3b-1}{2}$

31. $\frac{x-7}{5} + 2 = \frac{x+8}{10}$

32. $\frac{x-2}{4} - \frac{x-4}{6} = \frac{2}{3}$

33. $\frac{a-1}{3} + 3 = \frac{a+14}{9}$

34. $\frac{x+9}{9} + \frac{1}{3} = \frac{x-7}{2} - 1$

35. $\frac{5+x}{6} - \frac{10-x}{3} = 1$

36. $\frac{1}{3}(x+5) - 4 = \frac{x-8}{4} - \frac{1}{2}$

37. $\frac{x+11}{6} - \frac{10-x}{3} + 1 = 0$

38. $\frac{x+18}{4} - \frac{3}{7}(x-3) = 4$

39. $\frac{1}{10}(7b+1) - \frac{1}{4}(b-9) = 1$

40. $\frac{4a+1}{2} - \frac{1}{3}(2a+3) = \frac{5a-1}{4}$

41. $\frac{2}{x} = 3$

42. $\frac{3}{a} = -4$

43. $\frac{2}{3x} = \frac{1}{9}$

44. $\frac{5}{n} - \frac{1}{2} = 2$

45. $\frac{4}{x} + \frac{3}{2x} = \frac{11}{6}$

46. $\frac{2n+1}{2n} - \frac{3n-2}{3n} = \frac{7}{12}$

Solve these equations:

47. $\dfrac{a}{a+5} = \dfrac{1}{2}$

48. $\dfrac{3}{2} = \dfrac{x}{x+2}$

49. $\dfrac{b-2}{b+3} = \dfrac{3}{8}$

50. $\dfrac{x-7}{x+2} = \dfrac{1}{4}$

51. $\dfrac{5}{a+3} = \dfrac{2}{a}$

52. $\dfrac{6}{5y} = \dfrac{2}{y+4}$

53. $\dfrac{a-1}{a-2} = 1.5$

54. $\dfrac{y}{y-3} = \dfrac{6}{3}$

55. $\dfrac{2}{x+3} = \dfrac{5}{x}$

56. $\dfrac{-3}{x+5} = \dfrac{1}{2}$

57. $\dfrac{x}{3} + \dfrac{2x^2}{3x-4} = \dfrac{9x-2}{9}$

58. $\dfrac{2x-1}{2} - \dfrac{x+2}{2x+5} = \dfrac{6x-5}{6}$

59. $\dfrac{7}{x-4} = \dfrac{5}{x+2}$

60. $\dfrac{10}{x-3} = \dfrac{9}{x-5}$

61. $\dfrac{y+5}{y-3} + \dfrac{4}{y-3} = 5$

62. $\dfrac{x-1}{x+1} = \dfrac{x+3}{x+10}$

63. $\dfrac{x-3}{x+5} = \dfrac{x-2}{x+2}$

Literal Equations Containing Fractions

In solving the equations below you will encounter fractions involving letters, instead of numerical values. These equations may be solved by the same methods that are employed in the solution of numerical fractional equations.

(1) In solving each of the following equations for n, do you add, subtract, multiply, or divide? State how you know which process to use. Solve each of the equations for n:

(*a*) $an = b$

(*b*) $n + a = b$

(*c*) $n - a = b$

(*d*) $\dfrac{n}{a} = b$

(*e*) $n(a + b) = c$

(*f*) $\dfrac{n}{a+b} = c$

(2) Solve for n: $\dfrac{n+1}{c} + \dfrac{nb}{ac} = \dfrac{b}{ac} + \dfrac{n+1}{a}$

To clear of fractions, multiply both sides of the equation by ac:

$$a(n+1) + nb = b + c(n+1)$$

Perform the multiplications indicated:

$$an + a + nb = b + cn + c$$

To get all terms containing n on the left side and all the other terms on the right side, add $-cn$ and $-a$ to both sides:

$$an + bn - cn = b + c - a$$

To combine the n's, factor the left side:

$$n(a + b - c) = b + c - a$$

Divide both sides by $a + b - c$:

$$n = \frac{b + c - a}{a + b - c}$$

Can this answer be reduced?

Now that you know how to solve *for the value of one unknown* in fractional equations of the first degree, it is possible to summarize the *general procedure for solving all equations of the first degree for one unknown.*

The following are the steps in the procedure:

1. **Clear of fractions, if any.**
2. **Perform the multiplications, if any are indicated.**
3. **Combine terms if possible.**
4. **Get all the terms containing the unknown on the left side of the equation and all other terms on the right side.**
5. **In literal equations, if more than one term now contains the unknown, factor the side containing the unknown. In numerical equations combine the terms containing the unknown into one term.**
6. **Divide both sides of the equation by the coefficient of the unknown.**

You do not need to memorize this summary. Refer to it when necessary. In most cases it should be obvious to you what to do at each step.

Exercises

Solve the following equations for n:

1. $2n + a = b$
2. $2n - a = b$
3. $an - b = c$
4. $n(a - b) = c$
5. $n(a + b) + c = d$
6. $n(a + b) - c = d$
7. $an + bn = c$
8. $an - bn = c$
9. $an - bn + cn = d$
10. $an + bn + c = d$
11. $an + bn - c = d$
12. $an + b - cn = d$
13. $an - b + cn = d$
14. $an = c + bn$
15. $an = c - bn$
16. $an + c = b - dn$
17. $an - c = b + dn$
18. $an + b - c = d - bn$
19. $\frac{n}{3} + a = b$
20. $\frac{n}{3} + a = n$
21. $\frac{n}{a} = b$
22. $\frac{n}{a} = \frac{b}{c}$
23. $\frac{n}{a} + \frac{b}{c} = d$
24. $\frac{a}{c} = n + bn$
25. $\frac{n}{a + b} = c$
26. $\frac{n}{a + b} = c - n$
27. $r = 6 + 2(n - 1)$
28. $a = \frac{5}{9}(n - 32)$
29. $\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$
30. $\frac{1}{n} = \frac{1}{a} - \frac{1}{b}$

Solve for the required letter:

31. $i = prt$. Solve for r.
32. $s = \frac{W}{L}$. Solve for L.
33. $M = \frac{bh^3}{3}$. Solve for b.
34. $v = \frac{S}{T}$. Solve for T.
35. $s = \frac{ah}{r}$. Solve for a.
36. $v = \frac{bh^3}{3}$. Solve for b.
37. $Q = \frac{\pi r^2 v}{h}$. Solve for v.
38. $c = \frac{E}{R}$. Solve for R.
39. $I = \frac{bd^3}{3}$. Solve for b.
40. $A = \frac{F}{M}$. Solve for M.
41. $M = \frac{SI}{T}$. Solve for S.
42. $R = \frac{EI}{M}$. Solve for E.

Solve for the required letter:

43. $C = \frac{Ka - b}{a}$. Solve for K, for a, and for b.

44. $r = \frac{v^2 pL}{a}$. Solve for p and for a.

45. $R = \frac{WL - x}{L}$. Solve for W, for L, and for x.

46. $L = \frac{Mt - g}{t}$. Solve for t and for g.

47. $v = \pi L r^2$. Solve for L.

48. $f = \frac{gm - t}{m}$. Solve for g, for m, and for t.

49. $S = \frac{rl - a}{r - 1}$. Solve for r, for l, and for a.

50. $R + nr = \frac{En}{c}$. Solve for R, r, n, and c.

51. $v_1 = v_2 - \frac{n}{t}$. Solve for v_2, t, and n.

52. $\frac{W}{2n} = \frac{E}{n - r}$. Solve for n, E, and r.

53. Assume that all letters in the formula $f = \frac{gm - t}{m}$ represent positive numbers. If the value of m is increased and g and t remain fixed, is the value of f *increased* or *decreased?* What happens if t is negative and m is increased? Give evidence to support your answer.

54. The formula for the coefficient of lift, used in aerodynamics, is $C_L = \frac{L}{\frac{\rho}{2} S V^2}$. Solve for S. Then find the value of S when $L = 864$, $\rho = .003$, $C_L = .1$, and $V = 120$. (You will learn the meaning of these letters when you study aeronautics.)

55. The formula $W = C_L \frac{\rho}{2} S V^2$ can be changed so as to find the velocity needed to sustain the weight of a plane under certain conditions. Solve for V^2. Then find V when $W = 1500$, $C_L = .43$, $\rho = .002$, and $S = 200$.

General Problems

Problems may be *particular* or *general*. Here is an example of a particular problem and its solution.

The sum of two numbers is 20, and the larger is 9 times the smaller. Find the numbers.

The smaller number	x
The larger number	$9x$

$$x + 9x = 20$$
$$10x = 20$$
$$x = 2$$

The two numbers are 2 and 18.

Next we shall have a general problem.

The sum of two numbers is s, and the larger is m times the smaller. Find the numbers.

The smaller number	x
The larger number	mx

$$x + mx = s$$
$$x(1 + m) = s$$
$$x = \frac{s}{1 + m} \qquad mx = \frac{ms}{1 + m}$$

Note that the particular problem solved above is a special case of the general problem, for in it $s = 20$ and $m = 9$. The expressions $\frac{s}{1+m}$ and $\frac{ms}{1+m}$ may be regarded as general expressions, or formulas, for finding two numbers when their sum and quotient are known. What are the numbers if $s = 30$ and $m = 4$? if $s = 100$ and $m = 9$?

In the following exercises you will first solve a particular problem, with arithmetical numbers, and then, with literal numbers, solve a general problem which represents all the special problems of that type.

Exercises

1. (*a*) Particular problem: The sum of two numbers is 16; the larger is 4 more than the smaller. Find each number.

(*b*) General problem: The sum of two numbers is s; the larger is a more than the smaller. Solve for the two numbers; i.e., make a formula for finding each number in this type of problem. Find the two numbers when s is 40 and a is 10.

2. (*a*) The sum of two numbers is 32, and their difference is 8. Find each number.

(*b*) The sum of two numbers is s, and their difference is d. Find each number.

(*c*) Read the formulas you obtain in (*b*) as rules for finding the numbers.

(*d*) Find the two numbers when $s = 52$ and $d = 10$.

3. (*a*) The sum of two numbers is 9; 10 times the smaller equals 5 times the larger. Find the numbers.

(*b*) The sum of two numbers is s, and m times the smaller equals n times the larger. Find each number.

(*c*) Read the formulas in (*b*) as rules for finding the numbers.

(*d*) Find the two numbers when $s = 12$, $m = 6$, and $n = 2$.

4. (*a*) A rectangle is 10 ft. longer than it is wide; its perimeter is 72 ft. Find its length and width.

(*b*) A rectangle is b ft. longer than it is wide; its perimeter is p ft. Find its length and width.

(*c*) Read the results in (*b*) as rules for solving any problem of this type.

(*d*) Find the length and the width when $b = 8$ and $p = 96$.

(*e*) Make up a particular problem and solve it by the use of these formulas.

5. (*a*) The sum of two numbers is 14; 3 times their sum is equal to 21 times their difference. Find each number.

(*b*) The sum of two numbers is s; m times their sum is equal to n times their difference. Find each number.

(*c*) Read the formulas in (*b*) as rules for solving problems of this type.

(*d*) Make up a particular problem which you can solve by these formulas.

(*e*) In problems of this type, can $m = n$? Can m be greater than n? Can it be smaller than n?

6. (*a*) Separate 20 into two parts, such that the quotient of the larger by the smaller shall be $\frac{3}{2}$. Use x for the smaller.

(*b*) Separate n into two parts, such that the quotient of the larger by the smaller shall be $\frac{a}{b}$.

(*c*) Read the formulas which you obtained in (*b*) as rules for solving problems of this type.

(*d*) Make a particular problem which belongs to this type.

(*e*) Can $a = b$? Can a be smaller than b? Why?

7. (*a*) Think of some number; multiply it by 3; add 9 to the result; multiply the last result by 2; divide the last result by 6; subtract the number thought of in the beginning; the result must be 3. Can you tell why?

(*b*) Think of some number, as n; multiply it by a; then add $3a$; then multiply by 2; then divide by $2a$; then subtract the number thought of in the beginning. Show that the result must be 3. Can you make up other problems similar to this?

8. (*a*) Find the cost of a rectangular piece of land 80 rods wide and 120 rods long, at $120 an acre.

(*b*) Make a formula for solving any problem of the type given in (*a*), using l and w for the dimensions of the piece of land in rods, c for the cost, and n for the price per acre.

Verbal Problems Involving Fractions

1. The sum of two fractions is $\frac{11}{16}$, and one of them is known to be $\frac{5}{6}$ of the other. Find the fractions.

Smaller fraction	$\frac{5}{6}n$
Larger fraction	n

$$n + \frac{5n}{6} = \frac{11}{16}$$

2. The difference between two fractions is $\frac{1}{8}$. One fraction is $\frac{2}{3}$ of the other. Find the fractions.

3. Separate 120 into two parts so that one part is $\frac{3}{5}$ of the other part.

4. Separate 95 into two parts so that one part is $\frac{2}{3}$ of the other.

5. One number is 4 more than another. The quotient of the larger by the smaller is $\frac{5}{2}$. Find the numbers.

6. One number is 7 less than another. The quotient of the larger by the smaller is $\frac{4}{3}$. Find the numbers.

7. Separate 72 into two parts so that their quotient is $\frac{2}{3}$.

8. What number added to both numerator and denominator of the fraction $\frac{3}{5}$ will make the value of the resulting fraction $\frac{3}{4}$?

9. Two numbers are consecutive (n and $n + 1$). If 6 is added to the first and 2 is subtracted from the second, the quotient of the resulting numbers is $4\frac{1}{2}$. Find the numbers.

10. What number must be added to the numerator and subtracted from the denominator of the fraction $\frac{17}{32}$ to make a fraction equal to $\frac{3}{4}$?

11. A is $\frac{5}{6}$ as old as B. Five years ago he was $\frac{4}{5}$ as old as B. How old is A?

12. The numerator of a fraction is 3 smaller than its denominator; if 2 is added to the numerator, the value of the fraction becomes $\frac{3}{4}$. What is the fraction?

13. The numerator of a fraction exceeds its denominator by 3. If 1 is added to the denominator and 5 is subtracted from the numerator, the value of the fraction becomes $\frac{1}{2}$. What is the fraction?

14. The numerator of a fraction is 8 less than the denominator. If 3 is added to the numerator and 1 is subtracted from the denominator, the resulting fraction is $\frac{3}{5}$. What is the fraction?

15. An auto tourist made a trip of 120 miles, at a certain rate. On the return trip he doubled his rate, and required 3 hours' less time. Find his rate going. (Part of the solution follows.)

Rate going	r
Rate returning	$2r$
Time going	$\frac{120}{r}$
Time returning	?

Complete the solution of this problem.

Fractions in Work Problems

A teacher asked the following question: "If Arthur can sweep the snow from a sidewalk in 6 minutes and Jack can do it in 8 minutes, how long ought it to take them working together?"

Frances answered: "It would take them 14 minutes."

Alice said: "I'd find the average. It would take them 7 minutes."

Donald said: "I think both answers are wrong."

What do you think?

The following exercises will help you to answer the question.

(1) If Arthur can sweep the walk in 6 minutes, what part of the work can he do in 1 minute?

(2) If Jack can sweep the walk in 8 minutes, what part of the work can he do in 1 minute?

(3) What part of the work can they both do in 1 minute, working together?

(4) If both together can sweep the walk in n minutes, what part of the work will they do in 1 minute?

Solution of Problem

Number of minutes for Arthur to do the work	6
Number of minutes for Jack to do the work	8
Number of minutes for both to do the work	n
Amount of work done by Arthur in 1 minute	$\frac{1}{6}$
Amount of work done by Jack in 1 minute	$\frac{1}{8}$
Amount of work done by both in 1 minute	$\frac{1}{n}$

$$\frac{1}{6} + \frac{1}{8} = \frac{1}{n}$$
$$4n + 3n = 24$$
$$7n = 24$$
$$n = 3\tfrac{3}{7}$$

The boys ought to sweep the walk in $3\frac{3}{7}$ minutes.

Exercises

1. Farmer Brown can plow one of his fields with a tractor in 3 days. It would take his neighbor 15 days to plow the same field with a team of horses. How long will it take Mr. Brown if his neighbor helps him with the team?

2. One man can lay a sidewalk in 4 days, and another can do it in 4.5 days. How long does it take them working together?

3. A large pipe can empty a tank in 5 minutes, and a smaller pipe can empty it in 8 minutes. How long would it take to empty the tank if both pipes were draining the tank?

4. James can drive his car over a route in 5 hours, and Edgar can drive his car over the same route in 4 hours. How long would it take them to meet if they started at opposite ends at the same time?

5. A can do a piece of work in 3 days, B in 5 days, and C in 8 days. If all three work together, how many days will be required to do the work?

6. A can do a piece of work alone in 8 days. After working alone for 2 days he is joined by B and together they finish the work in 2 more days. How long would it take B alone to do the work? (Part of the solution follows.)

Number of days for B to do the work	n
Amount of work done by A in 1 day	
Amount of work done by B in 1 day	$\frac{1}{n}$
Amount of work done by A in 4 days	
Amount of work done by B in 2 days	

Note that the amount of work done by A in 4 days, plus the amount done by B in 2 days, is the whole work; that is, $\frac{5}{5}$ or $\frac{6}{6}$ or $\frac{8}{8}$, all of which equal 1.

7. One machine can complete an order for screws in 7 hours and another machine in 5 hours. How long will it take both machines to finish the job after the slower machine has been working alone for $3\frac{1}{2}$ hours?

8. Is the following problem sensible? If John can run 100 yards in 10 seconds and Frank can run 100 yards in 12 seconds, how long will it take them to run 100 yards together?

Problems Dealing with Mixtures

In practical life many problems that deal with mixtures arise, and many of these are fractional problems. On this and the next page you will learn how these problems can be solved.

(1) How many pounds of water must be added to 20 pounds of a 5% solution of salt and water to make it a 4% solution? A "5% solution of salt and water" means that 5% of the solution is salt.

Number of pounds of water to be added	n
Number of pounds of the 4% solution	$n + 20$
Number of pounds of salt in the 5% solution	$.05(20)$
Number of pounds of salt in the 4% solution	$.04(n + 20)$

The important thing for you to see here is that no salt has been taken out or put in; hence the amount of salt in the second solution is equal to the amount of salt in the first solution.

$$\begin{aligned} .05(20) &= .04(n + 20) \\ 1.00 &= .04\,n + .80 \\ 100 &= 4\,n + 80 \\ 20 &= 4\,n \\ n &= 5 \end{aligned}$$

Hence 5 pounds of water must be added.

(2) Dr. Altman has a 10% solution of iodine weighing 10 oz. How much alcohol must he add to make it a 5% solution?

Number of ounces of alcohol to be added	n
Number of ounces in the new solution	
Number of ounces of iodine in the first solution	
Number of ounces of iodine in the new solution	

Note that the amount of iodine in the first solution is the same as the amount of iodine in the second solution.

Exercises

1. If a patent medicine contains 25% alcohol, how much liquid other than alcohol should be added to 120 qt. of it to reduce it so that it will contain only 20% alcohol? How much liquid would have to be added so that the mixture will contain only 10% alcohol?

2. How much liquid must evaporate from 100 lb. of a 6% solution of water and salt to leave a residue 10% of which is salt?

3. In 40 oz. of alloy for watchcases there are 15 oz. of gold. How much copper must be added to the alloy so that a watchcase made from the new alloy weighing 2 oz. will contain $\frac{1}{2}$ oz. of gold?

4. In an alloy of silver and gold weighing 40 oz. there are 5 oz. of gold. How much silver must be added so that 5 oz. of the new alloy shall contain only $\frac{1}{2}$ oz. of gold?

5. Gun metal is made of tin and copper. An alloy of 1025 lb. of gun metal of a certain grade contains 861 lb. of copper. How much tin must be added so that 525 lb. of the new-grade gun metal will contain 430.5 lb. of copper?

Fractions in Solution of Simultaneous Equations

You have solved simultaneous equations like those in the exercises below by the addition method. They can be solved by the much more generally useful substitution method also. The solution of simultaneous equations by this method involves fractions, and you should become skillful in dealing with them. The illustrative problem below shows how the fractions are handled in cases of this kind.

(1) Solve
$$5x - 7y = 13$$
$$8y - 3x = -23$$

Solve for the letter with the smallest numerical coefficient; that is, for x in the second equation.

Subtracting $8y$, $\quad -3x = -23 - 8y$

Multiplying by -1, $\quad 3x = 23 + 8y$

Dividing by 3, $\quad x = \dfrac{23 + 8y}{3}$

Substitute this value of x in the other equation.

$$5\left(\frac{23 + 8y}{3}\right) - 7y = 13$$

Multiply each side of this equation by 3. By this process the denominator 3 will disappear.

$$\begin{aligned} 5(23 + 8y) - 21y &= 39 \\ 115 + 40y - 21y &= 39 \\ 19y &= -76 \\ y &= -4 \end{aligned}$$

Substitute -4 for y in the equation $x = \dfrac{23 + 8y}{3}$ and get $x = -3$.

Answer: $x = -3$, $y = -4$.

Exercises

Solve by the substitution method and check the following simultaneous equations:

1. $2x + 3y = 17$
   $3x - 2y = 6$
2. $2x - 3y = -18$
   $4x - 3y = -24$
3. $4x + 3y = 10$
   $3x + 2y = 5$
4. $-2x - 3y = 15$
   $3x - 5y = 63$
5. $6x + y = 2$
   $8x - 2y = 1$
6. $5x + 6y = 17$
   $6x + 5y = 16$
7. $3y + 2x = -1$
   $2y + 5x = 3$
8. $6p - 9q = 19$
   $15p + 7q = -41$
9. $7a + 8b = 19$
   $9a - 6b = 57$
10. $5y + 8p = 5$
    $3p - 2y = 29$

Chapter Summary

The study theme in this chapter has been the solving of equations that contain fractions. The general rule you have learned for the handling of such equations is to *clear them of fractions* and then proceed as in the solution of other equations. You have had much practice in solving fractional equations and have seen that in the construction of formulas and the solution of many problems fractions are involved. The important points for you to remember are (1) that multiplying both sides of an equation by the same number does not destroy the equality, (2) that by the use of multiplication an equation can be cleared of fractions, and (3) the number that should be used to multiply both sides of a fractional equation in order to clear it of fractions is the smallest number that will contain the denominators of all the fractions involved.

Chapter Review

Solve and check the following equations:

1. $\frac{n}{3} + \frac{n}{5} = \frac{16}{15}$

2. $\frac{3n + 5}{6} = \frac{n - 3}{3}$

3. $\frac{9}{x + 3} = \frac{7}{x - 2}$

4. $\frac{n - 3}{2} + \frac{2n + 5}{4} = \frac{n}{8}$

5. $\frac{n + 7}{6} - \frac{3n - 7}{8} + \frac{5n - 1}{12} = 3$

Solve for the required letter:

6. $A = \frac{bh}{2}$, for h

7. $E = \frac{17v^2}{2}$, for v

8. $P = \frac{A}{1 + rt}$, for A and for r

9. $S = \frac{n(a + l)}{2}$, for n and for a

10. $M = \frac{NR}{R - N}$, for N and for R

Solve for x and y:

11. $\frac{x}{4} + \frac{y}{5} = 1$
    $\frac{2x}{9} + 2 = \frac{y}{9}$

12. $\frac{1}{2}(x + 10) - y = -4$
    $\frac{3}{4}x - \frac{2(y + 2)}{4} = -10$

13. A man can do a piece of work in m hours, and a boy can do it in b hours. Write a formula for finding the number of hours it would take if both were working together.

CHAPTER XVI

RATIO AND PROPORTION

"Ratio" is a word frequently found in newspaper and magazine articles. One set of headlines read:

U. S. ARMY PLANES MORE
THAN MATCH FOR ENEMY

Destroying Foe's Aircraft
at Ratio of 7.5 to 1

Do you know what these headlines mean? When this question was asked in a class beginning the seventh grade, the answers showed little knowledge of the meaning. In a ninth-grade class the first answer was: "They mean that for every U. S. plane destroyed, $7\frac{1}{2}$ enemy planes were destroyed." That is correct, but when the students were asked how we could destroy one half a plane, they were bewildered. One boy said: "Perhaps we shot down 8 planes and one was only half destroyed." Was that a good answer?

In an eleventh-grade class there was no confusion. One student in this class said: "It might be easier to understand if we said that for every 2 of our planes destroyed, 15 enemy planes were destroyed. That is the same ratio. A ratio does not tell the actual number of things; it tells what part one number is of another or how many times another number it is. 1 to 7.5, 2 to 15, 4 to 30, and so on, all indicate the same ratio." His statements were all correct. Do you understand them?

Assuming that 45 enemy planes were destroyed, how many U. S. planes were lost?

Meaning of Ratio

When you say that you had twice as many exercises correct today as yesterday, or three fourths as many, you are stating the ratio of today's correct exercises to yesterday's correct exercises. In the first case the ratio is 2 to 1; in the second case it is 3 to 4. The ratio 2 to 1 means twice as many. The ratio 3 to 4 means three fourths as many. Does the statement tell in either case how many exercises you had right today? What would you have to know before you could tell?

The ratio of one number to another is the quotient obtained by dividing the first number by the second.

The ratio of a to b is $\frac{a}{b}$, sometimes written $a : b$. The ratio $\frac{a}{b}$ shows you what part a is of b, or what you must multiply b by to get a. Thus, the ratio of 40 to 120 is $\frac{40}{120}$, or $\frac{1}{3}$. This ratio shows you what part 40 is of 120; that is, 40 is $\frac{1}{3}$ of 120.

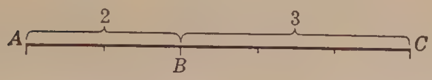
Ratios are fractions and all principles applying to fractions apply to ratios.

To express the ratio of two numbers, you must be sure that they represent quantities of the same kind and that they are measured in the same units. Thus, the ratio of 5 inches to 6 quarts has no meaning, since the quantities are of different kinds and are measured in different units. Also, the ratio of 6 inches to 2 feet is not 6 to 2. Before finding this ratio, you should change 2 feet to 24 inches and get the ratio 6 to 24.

In the first five of the exercises on the next page, answer the questions by referring to the line at the top of the page. This line has been divided into five equal parts. These and the additional exercises that follow should help you to review what you have already learned about ratios.

(1) Write as fractions reduced to lowest terms the ratios of these pairs of measures:

(*a*) \$40, \$100
(*b*) 35°, 180°
(*c*) 3 ft., 3 yd.
(*d*) 1000 lb., 2 tons
(*e*) 3 a ft., 12 a ft.
(*f*) 12 oz., 3 lb.



(2) What is the ratio of AB to BC? of BC to AB? of AB to AC? of BC to AC? of AC to BC? of AC to AB?

(3) AB is what part of BC? AB is what part of AC? BC is what part of AC?

(4) If BC is 24 inches, how long is AB?

(5) If AB is 12 inches, how long is BC?

(6) If AC is 35 inches, how long are AB and BC?

(7) If S is a point on a line RT between R and T, RS is 3 inches, and ST is 5 inches, what is the ratio of RS to ST? of RS to RT? of ST to RT? of RT to ST?

(8) Give examples of several pairs of numbers that are in the ratio $1:3$; $2:1$; $2:3$; $3:2$.

Exercises

1. In the ninth-grade class in Longmeadow there are 36 boys and 45 girls. What is the ratio of the number of boys to the number of girls? of the number of girls to the number of boys? of the number of boys to the total number of pupils?

2. At a picnic there were 15 adults and 60 children. What was the ratio of the number of adults to the number of children? of the number of children to the total number of persons present?

3. If the ratio of the number of U. S. planes destroyed to the number of enemy planes destroyed is $2:15$, what part of the total number of these destroyed planes belong to the United States?

4. Write the following ratios as fractions and reduce them to lowest terms:

(*a*) 4 to 8

(*b*) 6 to 24

(*c*) $15:25$

(*d*) 12 to 18

(*e*) $2x:3x$

(*f*) $24a^2:36a^2$

5. Give the ratios of the following:

(*a*) 40 minutes to 1 hour
(*b*) 8 days to 2 weeks
(*c*) 30 cents to \$1.50
(*d*) 1 sq. yd. to 1 sq. ft.

6. Separate 72 into two parts which are in the ratio 4 to 5.

First method: Represent the smaller part by x and the other by $72 - x$. Then $\frac{x}{72 - x} = \frac{4}{5}$.

Second method: Represent the smaller part by $4x$ and the larger part by $5x$. Then $4x + 5x = 72$.

7. Separate 99 into two parts which are in the ratio 4 : 7.

8. Henry and his younger brother decided to divide the profit from their small garden in the ratio 5 : 3. If the profit for one summer was \$12, how should it be divided?

9. If the ratio of cruising speed to maximum speed of an airplane is 5 to 6 and the maximum speed is 390 miles an hour, what is the cruising speed? If the cruising speed is 300 miles an hour, what is the maximum speed?

Ratios Expressed as Decimals

Ratios are frequently expressed as decimals. Thus the ratio of 3 : 6 is .50 and the ratio of 12 : 16 is .75. The ratio of 1 : 3 is .33 correct to the nearest hundredth. These decimals are obtained by dividing the first number by the second.

(1) Express the following ratios as decimals correct to the nearest hundredth: (*a*) $\frac{1}{2}$ (*b*) $\frac{3}{5}$ (*c*) $\frac{6}{5}$ (*d*) $\frac{2}{3}$ (*e*) $\frac{17}{30}$

(2) If one line is 11 in. and another is 16 in., what is the ratio of the shorter to the longer expressed decimally to the nearest hundredth?

Exercises

In this set of exercises express all ratios as decimals correct to the nearest hundredth:

1. In an algebra class of 30 pupils, 6 pupils received honor grades. What was the ratio of the honor pupils to the total number of pupils?

2. During one month in the Park Valley School, 18 of the 72 pupils were absent at least one day. What was the ratio of the number of pupils having absences during the month to the total number of pupils?

Express the following ratios as decimals correct to the nearest hundredth:

3. $\frac{7}{10}$	6. $\frac{1}{3}$	9. $\frac{7}{13}$	12. $\frac{7.5}{10}$
4. $\frac{4}{10}$	7. $\frac{2}{3}$	10. $\frac{17}{21}$	13. $\frac{8.3}{10}$
5. $\frac{4}{5}$	8. $\frac{5}{6}$	11. $\frac{24}{83}$	14. $\frac{12.5}{10}$

15. If one line is 8 in. long and another is 12 in., what is the ratio of the shorter to the longer?

16. If one line is 7.3 in. long and another is 10 in., what is the ratio of the shorter to the longer?

17. If the ratio of a to b is .35, what is a when b is 24?

18. If the ratio of a to b is .84, what is b (to the nearest tenth) when a is 61?

Proportion

In reducing ratios to their lowest terms, you have written such statements as $\frac{4}{8} = \frac{1}{2}$. This is an equation stating that two ratios are equal; it is called a *proportion*.

A proportion is an equation stating that two ratios are equal.

The equations $\frac{6}{8} = \frac{12}{16}$ and $\frac{a}{b} = \frac{c}{d}$ are proportions.

These proportions may also be written as $6:8 = 12:16$ and $a:b = c:d$. They are read: "6 is to 8 as 12 is to 16" and "a is to b as c is to d."

The four numbers in a proportion are called the *terms of the proportion.* In the proportion $\frac{a}{b} = \frac{c}{d}$, a, b, c, and d are the terms. The terms of a proportion are said to be *in proportion.*

Ratios are sometimes *inversely proportional;* that is, one ratio is equal to the reciprocal of the second ratio. If the ratio $\frac{a}{b}$ is *directly* proportional to the ratio $\frac{c}{d}$, you have the proportion $\frac{a}{b} = \frac{c}{d}$. But if the ratio $\frac{a}{b}$ is *inversely* proportional to the ratio $\frac{c}{d}$, you have the proportion $\frac{a}{b} = \frac{d}{c}$.

A proportion is a fractional equation and if one of the four terms is unknown, you can solve it as in any fractional equation. But since a proportion is a special kind of fractional equation, it can be solved by a special rule. Note the way in which the following proportions are cleared of fractions:

$\frac{1}{2} = \frac{3}{6}$. Note that $1 \times 6 = 2 \times 3$.

$\frac{3}{4} = \frac{9}{12}$. Note that $3 \times 12 = 9 \times 4$.

$\frac{5}{8} = \frac{20}{32}$. Note that $5 \times 32 = 8 \times 20$.

In each of the three examples above the so-called ***cross products*** are equal. This can be proved in general.

In the proportion $\frac{a}{b} = \frac{c}{d}$, you can write down the cross products ad and bc. You want to show that $ad = bc$. You can clear the equation $\frac{a}{b} = \frac{c}{d}$ of fractions by multiplying both sides first by b and then by d, or you can multiply both sides directly by the product bd. Doing the latter, we have $bd\left(\frac{a}{b}\right) = bd\left(\frac{c}{d}\right)$. This simplifies to $ad = bc$, from which you know that in general the cross products in a proportion are equal. This gives you a very simple way of solving an equation that is a proportion.

EXAMPLE 1. Solve $\frac{3}{4} = \frac{12}{x}$ for x.

SOLUTION. $\frac{3}{4} = \frac{12}{x}$

Getting cross products, $3x = 48$

Dividing by 3, $x = 16$

CHECK. $\frac{3}{4} = \frac{12}{16}$

EXAMPLE 2. A man earned \$669 in 3 months. At the same rate what would he earn in 7 months?

Note that the ratio of 669 to the unknown number must be equal to the ratio of 3 to 7. Write and solve the proportion $\frac{3}{7} = \frac{669}{x}$.

SOLUTION.

$$\frac{3}{7} = \frac{669}{x}$$
$$3x = 4683$$
$$x = 1561$$

He would earn \$1561.

CHECK.

$$\frac{3}{7} = \frac{669}{1561}$$

Exercises

Solve for n. (Give decimal answers correct to the nearest hundredth.)

1. $\frac{2}{3} = \frac{10}{n}$	7. $\frac{n}{a} = \frac{b}{c}$	13. $\frac{n}{2} = .7$
2. $\frac{n}{5} = \frac{3}{2}$	8. $\frac{a}{n} = \frac{b}{c}$	14. $\frac{n}{12} = .82$
3. $\frac{8}{n} = \frac{3}{7}$	9. $\frac{a}{b} = \frac{c}{n}$	15. $\frac{n}{150} = 1.45$
4. $\frac{5}{7} = \frac{n}{8}$	10. $\frac{12 - n}{n} = \frac{5}{7}$	16. $\frac{2}{n} = .3$
5. $\frac{n}{a} = \frac{5}{3}$	11. $\frac{n}{16 - n} = \frac{5}{3}$	17. $\frac{2}{n} = .37$
6. $\frac{n}{a} = \frac{b}{8}$	12. $\frac{n}{3 - n} = \frac{a}{b}$	18. $\frac{240}{n} = 1.62$

Solve the following problems by means of proportions:

19. If 3 oranges cost 15 cents, how much will 12 oranges cost? $\left(\frac{3}{12} = \frac{15}{x}.\right)$

20. If an automobile goes 110 miles in 3 hours, how far will it go in 5 hours at the same rate?

21. If 90 feet of wire weighs 18 pounds, what will 110 feet of the same kind of wire weigh?

22. If $1\frac{1}{2}$ inches on a map represents 60 miles, what distance does $2\frac{7}{8}$ inches on the map represent?

23. If I am making a scale drawing with $\frac{1}{2}$ inch representing a foot, how long a line will represent 2 inches?

24. What out-of-door distance is represented by a distance of $4\frac{7}{16}$ inches on a map if 60 miles is represented by $1\frac{1}{2}$ inches?

25. If y is proportional to x and y is 9 when x is 6, what is y when x is 10?

26. If y is proportional to x and y is 18 when x is 12, what is x when y is 24?

27. If y is proportional to x and $y = 100$ when $x = 5$, what is y when $x = 3$?

28. If y is proportional to x and y is 35 when x is 10, what is y when x is 7?

29. What is the weight of 150 feet of steel wire if 120 feet of the same kind weighs 36 pounds?

30. How much should I pay for 8 rods of iron fencing if 12.5 rods costs $100?

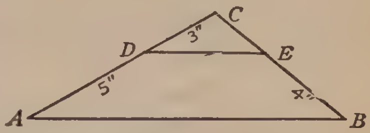
31. If 50 feet of fencing costs $300, how many feet can I buy for $425?

32. A certain object casts a shadow 10 feet long at the same time that another object 10 feet high casts a shadow 8 feet long. Find the height of the first object.

33. The number of days it takes to complete a certain piece of work in an airplane factory is inversely proportional to the number of men working. If it takes 18 days for 3 men to do it, how long will it take 9 men? ("Inversely proportional" in this problem means that the ratio of the number of men is equal to the reciprocal of the ratio of the corresponding number of days; that is, $\left(\frac{3}{9} = \frac{x}{18}\right)$.

34. When a gas in a container is placed under pressure, the volume of the gas is inversely proportional to the pressure. If the volume of gas is 12 cubic centimeters when the pressure is 8 pounds, what is the volume under a pressure of 15 pounds?

*35. The striking force of a moving object of a given weight, say an automobile in an accident, is proportional to the square of its speed. If the speed of a car is increased from 30 to 60 miles an hour, how many times is the striking force increased? If the speed is increased from n to $2n$ miles an hour, how many times is the striking force increased?

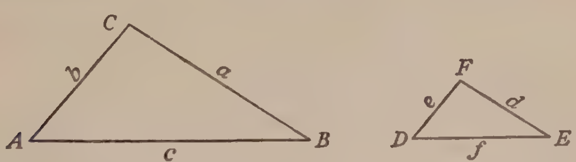


36. In geometry, we learn that if DE is parallel to AB in any triangle, then the four parts of AC and BC are always in proportion; that is, $\frac{AD}{DC} = \frac{BE}{EC}$. How long is CE if $AD = 5$ in., $DC = 3$ in., and $BE = 4$ in.? (Substitute 5 for AD, 3 for DC, and 4 for BE in the given proportion. Let EC be x.)

37. Using the figure for Ex. 36, find the length of DC when $AD = 7$ in., $BE = 4$ in., and $CE = 3$ in.

Proportions in Similar Triangles

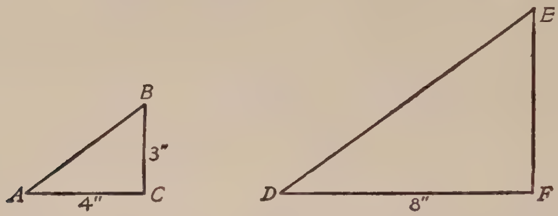
Triangles which have the same shape, but not necessarily the same size, are ***similar triangles***. In similar triangles the corresponding angles are equal (in the triangles below, angle A = angle D, angle B = angle E, and angle C = angle F) and the ratio of any two sides of one is equal to the ratio of the corresponding sides of the other $\left(\frac{a}{b} = \frac{d}{e}, \frac{a}{c} = \frac{d}{f}, \frac{b}{c} = \frac{e}{f}\right)$.



(1) Triangle ABC and triangle DEF above, are similar. If $a = 10$ in., $b = 7$ in., and $d = 5$ in., how long is e? $\left(\frac{10}{7} = \frac{5}{x}\right)$

(2) If in the similar triangles of Ex. 1 above, $b = 15$ in., $e = 10$ in., and $a = 12$ in., how long is d?

(3) The triangles below are similar. Hence $\frac{AC}{BC} = \frac{DF}{EF}$. If $AC = 4$ in., $BC = 3$ in., and $DF = 8$ in., how long is EF?



Importance of Ratio and Proportion

Ratio and proportion are used constantly in science and in everyday life. Ratios are employed in the calculation of human diets, and in determining the quantities of the ingredients that are used in mixing foods for animals and fertilizers for the soil. Ratios are strictly maintained among the different materials used in the manufacture of such products as soap, cement, glass, and steel. In machinery all levers and pulleys and gears and wheels are made with definite ratios to each other, so that the machines will give the power or have the speed desired. Many scientific laws are in terms of ratios. Indeed, four out of five problems in elementary physics and chemistry involve ratio and proportion, and in any advanced science or mechanical work a knowledge of these is indispensable.

Chapter Summary

The ratio of two numbers is found by indicating that the first number is to be divided by the second. It may be expressed as a fraction, but it is often written as a decimal found by dividing the numerator by the denominator.

A *proportion* is an equation stating that two ratios are equal. It can be solved by a special rule which says that *in any proportion the cross products are equal.* Many problems may be solved by setting up the conditions as a proportion. Both ratios and proportions have far-reaching applications in the world in which we live.

You should understand the following technical terms:

Ratio   Proportion   Proportional   Inversely proportional

Chapter Review

1. What is the ratio of 2 ft. to 30 in.?

2. Express the ratio 75 to 60 in its simplest form.

3. After 200 gal. of fuel oil have been used from a full 500-gal. tank, what is the ratio of the amount used to the amount left in the tank? the ratio of the amount left to the amount used? the ratio of the amount used to the capacity of the tank?

4. In the preceding exercise, what part of the total capacity of the tank had been used? What part of the total capacity was left? How did the amount used compare with the amount left?

5. If the 200 gal. of fuel oil was used in 30 days, how long will the remaining 300 gal. last? (Assume similar conditions.)

6. Separate 35 into two parts which shall have the ratio 2 to 3.

7. Express the ratio $\frac{12.3}{7}$ as a decimal correct to hundredths.

8. If the ratio of a to b is .62, what is a when b is 42?

9. If the ratio of a to b is .12, what is b to the nearest tenth when a is 32?

10. Solve the following proportions for n:

(a) $\frac{3}{4} = \frac{7}{n}$ (b) $\frac{n}{5} = \frac{3}{4}$ (c) $\frac{n}{3 - n} = \frac{5}{6}$

11. If y is proportional to x and y is 4 when x is 5, what is y when x is 6?

12. If y is inversely proportional to x and y is 45 when x is 3, what is y when x is 5?

13. A can travel 50 miles while B is traveling 40 miles. At the same rates, how far will B travel while A is going 42 miles? Give the result to the nearest tenth of a mile.

Finding the height of a cloud by trigonometry.

Chapter XVII

NUMERICAL TRIGONOMETRY

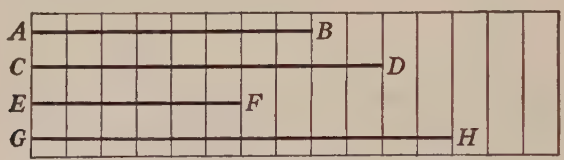
Numerical trigonometry is one of the most useful branches of elementary mathematics. Astronomers use trigonometry to measure distances to the sun, moon, and stars and to other planets. Surveyors use it to find the height of mountains they cannot climb and the width of rivers they cannot cross. It is used by engineers in many of their computations. It is indispensable in the navigation of seagoing vessels. Without numerical trigonometry it would be impossible for the navigator of an airliner to locate his position in crossing an ocean. These are only a few of the applications of numerical trigonometry. In this chapter you will be introduced to this important subject.

The word *trigonometry* is made up of two parts — *trigon* meaning triangle, and *metry* referring to measurement. Trigonometry is triangle measurement. It includes a study of the relations between the sides and the angles of triangles, and makes use of the fact that in two similar triangles the ratio of the lengths of any two sides of one is equal to the ratio of the lengths of the corresponding sides of the other. If we know a certain number of parts of a triangle, either sides or sides and angles, we can determine the other parts. In indicating angles the sign $\angle$ is used as the equivalent of the word "angle." $\angle A$ is read as "angle A."

Ratios of Lengths of Line

In numerical trigonometry, the ratios of lengths of sides of right triangles are used constantly. Since the sides of these triangles are straight lines, this means that you will be using ratios of lengths of lines very frequently.[1] It will help you in dealing with trigonometric ratios to practice using ratios of lengths of lines in the exercises that follow.

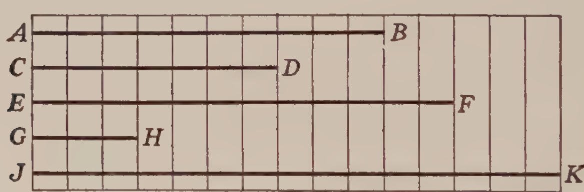
(1) If each unit space on the lines in the illustration below represents a foot, what are the lengths which *AB*, *CD*, *EF*, and *GH* represent?



(2) What is the ratio of the length of *AB* to the length of *CD*, written as a fraction reduced to lowest terms? of *CD* to *EF*? of *EF* to *GH*? of *CD* to *GH*? of *AB* to *EF*? of *GH* to *AB*?

(3) Express the value of the following ratios as decimals correct to hundredths (see the figure for Ex. (1) above):

(*a*) $\frac{AB}{CD}$, (*b*) $\frac{CD}{EF}$, (*c*) $\frac{EF}{GH}$, (*d*) $\frac{CD}{GH}$, (*e*) $\frac{AB}{EF}$, (*f*) $\frac{GH}{AB}$



(4) If each unit space in the figure above represents a foot, what are the lengths which *AB*, *CD*, *EF*, *GH*, and *JK* represent?

(5) Which of the following ratios (expressed as decimals) are correct? (See figure above.)

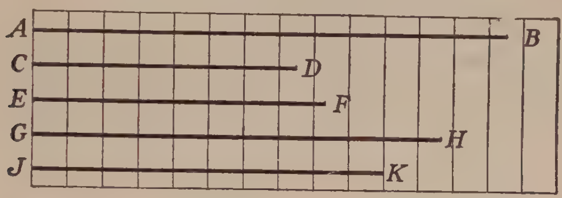
(*a*) $\frac{CD}{AB} = .7$, (*b*) $\frac{EF}{AB} = 1.2$, (*c*) $\frac{GH}{AB} = .4$, (*d*) $\frac{JK}{AB} = 1.5$

[1] In the discussions that follow, it will often be simpler to refer to *ratios of the lengths of any two sides* of a triangle as *ratios of the sides* without specifying that we mean lengths of the sides each time.

(6) If $AB = 10$ units and $CD = 2.4$ units, what is the value of the ratio $\frac{CD}{AB}$ expressed as a decimal?

(7) Verify the following statements about the lines in the figure below. If each small space is a unit —

(*a*) *AB* is about 13.5 units.
(*b*) *CD* is about 7.5 units.
(*c*) *EF* is about 8.3 units.
(*d*) *GH* is about 11.7 units.



(8) Find the ratio of each of these lengths to the length of *JK*. Express the ratios as decimals.

(9) If $a = 1.68b$ and b is 2.48 in. long, how long is a? (Give your answer to the nearest hundredth.)

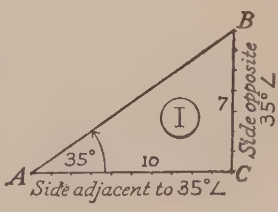
(10) If $a = 1.68b$ and $a = 2.48$, how long is b?

An Important Ratio in Right Triangles

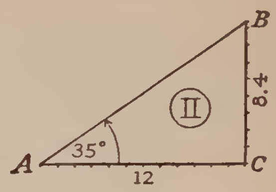
A *right triangle* is one that has a *right angle* (an angle of 90°). The other two angles are *acute angles* (less than 90°), since the sum of all three angles is always 180° in any triangle. In this section you will learn about an important ratio of the sides in a right triangle.

Right triangle

(1) What is the length of *BC*, the side opposite the 35° angle in triangle I? (Note the units of measurement marked off on *AC* and *BC*.) What is the length of *AC*, the side adjacent to the 35° angle? What is the ratio of *BC* to *AC*?

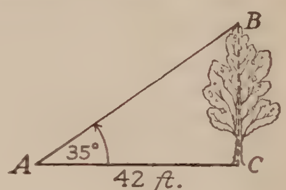


(2) In triangle II, BC is about 8.4 units. What is the ratio of BC to AC in triangle II? What is the ratio of *the side opposite the 35° angle* to *the side adjacent to the 35° angle* in triangle II?



In both Exs. (1) and (2) $\angle A$ is 35° and the ratio of the *side opposite this angle* to the *side adjacent to this angle* is .7. This is true of all right triangles with an acute angle of 35°. The reason for this is that all such triangles are similar. This fact enables us to determine the length of one of these two sides when the length of the other side is known (provided the angle is 35°).

(3) If the distance AC is 42 ft. and angle A is 35°, what is the height of the tree?



SOLUTION. $\dfrac{\text{Side opposite } 35°}{\text{Side adjacent to } 35°} = .7$

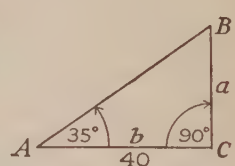
$$\frac{BC}{AC} = .7$$

Letting $BC = n$ and substituting, $\dfrac{n}{42} = .7$

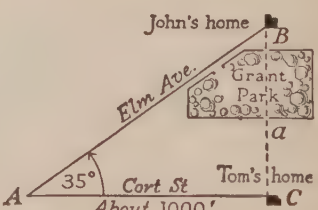
Solving for n, $n = 29.4$

ANSWER. The height of the tree is 29.4 ft.

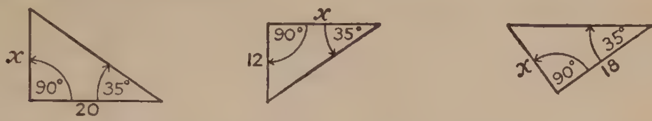
(4) The pupils of Parker School studied the figure at the right. One group said that $\dfrac{40}{a} = .7$ is a correct equation for the figure. The others wrote the equation: $\dfrac{a}{40} = .7$. Which result was correct? How do you decide?



(5) Study the figure at the right and write an equation from which the distance between John's and Tom's homes can be found. The angle at C is 90°. Find the distance.



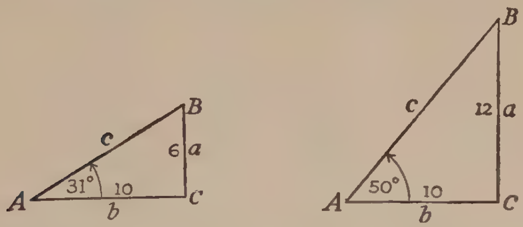
(6) Find the length of the side marked x in each of the triangles below. (The side opposite the given angle is the numerator of the ratio.)



(7) Using the figures below, answer the following questions:

(*a*) If $\angle A$ is 50°, is the ratio of $\frac{a}{b}$ more or less than .7?

(*b*) If $\angle A$ is 31°, is the ratio of $\frac{a}{b}$ more or less than .7?



For all right triangles large or small, having an angle of 35°, the ratio of the side opposite the 35° angle to the side adjacent to it is .7. As the size of the angle changes, however, the ratio also changes. As the angle increases, the ratio increases.

> **For any given acute angle in a right triangle, the ratio of *the side opposite to the side adjacent* is the same for all right triangles.**

The Tangent of an Angle

You have learned that the side opposite angle A in a right triangle divided by the side adjacent to angle A is always the same ratio for a given value of A, no matter what the size of the triangle is. This ratio is called the ***tangent of angle A.*** Thus, the tangent of 35° means the ratio of a (the side opposite 35°) to b (the side adjacent to 35°) in a right triangle.

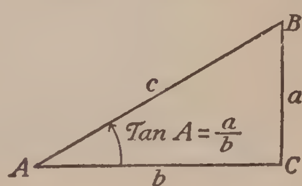


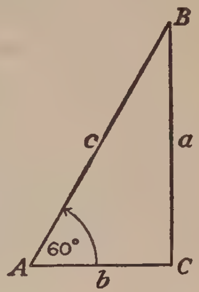
TABLE OF TANGENTS, COSINES, AND SINES

Angle	Tangent $\left(\frac{opp.}{adj.}\right)$	Cosine $\left(\frac{adj.}{hyp.}\right)$	Sine $\left(\frac{opp.}{hyp.}\right)$	Angle	Tangent $\left(\frac{opp.}{adj.}\right)$	Cosine $\left(\frac{adj.}{hyp.}\right)$	Sine $\left(\frac{opp.}{hyp.}\right)$
0°	.000	1.000	.000	45°	1.000	.707	.707
1	.017	1.000	.017	46	1.036	.695	.719
2	.035	.999	.035	47	1.072	.682	.731
3	.052	.999	.052	48	1.111	.669	.743
4	.070	.998	.070	49	1.150	.656	.755
5	.087	.996	.087	50	1.192	.643	.766
6	.105	.995	.105	51	1.235	.629	.777
7	.123	.993	.122	52	1.280	.616	.788
8	.141	.990	.139	53	1.327	.602	.799
9	.158	.988	.156	54	1.376	.588	.809
10	.176	.985	.174	55	1.428	.574	.819
11	.194	.982	.191	56	1.483	.559	.829
12	.213	.978	.208	57	1.540	.545	.839
13	.231	.974	.225	58	1.600	.530	.848
14	.249	.970	.242	59	1.664	.515	.857
15	.268	.966	.259	60	1.732	.500	.866
16	.287	.961	.276	61	1.804	.485	.875
17	.306	.956	.292	62	1.881	.469	.883
18	.325	.951	.309	63	1.963	.454	.891
19	.344	.946	.326	64	2.050	.438	.899
20	.364	.940	.342	65	2.145	.423	.906
21	.384	.934	.358	66	2.246	.407	.914
22	.404	.927	.375	67	2.356	.391	.921
23	.424	.921	.391	68	2.475	.375	.927
24	.445	.914	.407	69	2.605	.358	.934
25	.466	.906	.423	70	2.747	.342	.940
26	.488	.899	.438	71	2.904	.326	.946
27	.510	.891	.454	72	3.078	.309	.951
28	.532	.883	.469	73	3.271	.292	.956
29	.554	.875	.485	74	3.487	.276	.961
30	.577	.866	.500	75	3.732	.259	.966
31	.601	.857	.515	76	4.011	.242	.970
32	.625	.848	.530	77	4.331	.225	.974
33	.649	.839	.545	78	4.705	.208	.978
34	.675	.829	.559	79	5.145	.191	.982
35	.700	.819	.574	80	5.671	.174	.985
36	.727	.809	.588	81	6.314	.156	.988
37	.754	.799	.602	82	7.115	.139	.990
38	.781	.788	.616	83	8.144	.122	.993
39	.810	.777	.629	84	9.514	.105	.995
40	.839	.766	.643	85	11.430	.087	.996
41	.869	.755	.656	86	14.301	.070	.998
42	.900	.743	.669	87	19.081	.052	.999
43	.933	.731	.682	88	28.636	.035	.999
44	.966	.719	.695	89	57.290	.017	1.000
45	1.000	.707	.707	90		.000	1.000

We have found that the tangent of 35° = .7 (approx.). The tangent of 35° is often written as *tan 35°*.

According to this definition tan B is $\frac{b}{a}$ since in this case b is the side opposite the angle and a is the side adjacent to the angle. Tan B ($\angle B$ is 55°. Why?) will be more than .7 since the tangent increases as the size of the angle increases.

The following exercises will help you to understand what is meant by the tangent ratio.



(1) Using graph paper and a protractor, draw a right triangle as shown here, making b 10 units and angle A 60°. Compute the value of $\frac{a}{b}$. This ratio is, of course, tan 60°. (If you use a large piece of graph paper and let the side of 10 small squares represent one unit, your work will be more accurate.)

(2) Do the same thing for angles of 10°, 20°, 30°, 40°, 50°, and 70°. Check your accuracy by comparison with the table on page 412.

Finding the Tangents of Angles from a Table

When people need to use tangents, they read them from a table; they do not take the time to work them out as we have done. The tables are computed by means of higher mathematics and are therefore more accurate than they would be if found by measurement.

(1) Using the table on page 412, find tangent 62°. (In the column headed "Angle" find **62**. Opposite this number in the column headed "Tangent" find the number 1.881. Then, tan 62° = 1.881.)

(2) Using the table on page 412, find the angle whose tangent is .649. (In the column headed "Tangent" find the number .649. Opposite this number in the column headed "Angle" you will see the number **33**. Then, the angle whose tangent is .649 is 33°.

Exercises

1. Use the table of tangents to find the following: (*a*) tangent of 20°; tan 22°; (*b*) tangent of 40°; (*c*) tangent of 45°; tan 55°; (*d*) tangent of 70°; tan 73°.

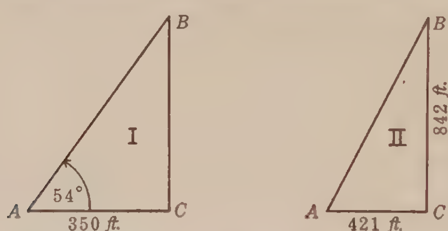
2. From the table of tangents find the tangent of 5°; of 25°; of 37°; of 54°; of 80°; of 85°; of 89°.

3. From the table of tangents find the angle whose tangent is .58; 1.73; .12; .60; 1.60; 2.25.

4. Refer to the table and complete: As the angle increases from 0° to 89°, the tangent of the angle _?_ from .000 to _?_.

5. Is the tangent of a 50° angle twice as great as the tangent of a 25° angle? Is the tangent of an 88° angle twice as great as the tangent of a 44° angle? Is the tangent of a 60° angle three times as great as that of a 20° angle?

6. Why is the tangent of a 90° angle not given in the table?



7. If in triangle I with the right angle at C, $\angle A$ is 54°, $\angle C$ is 90°, and AC is 350 ft., how long is BC to the nearest foot?

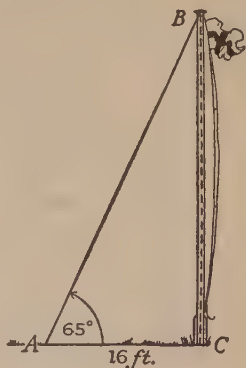
8. If in triangle II, $\angle C$ is 90°, BC is 842 ft., and AC is 421 ft., how large is $\angle A$ to the nearest degree?

Practical Use of the Tangent

In solving each of the following problems, you will need to use the tangent ratio in connection with a right triangle suggested by the problem. In each problem you will be given the number of degrees in one acute angle of the triangle and the length of a side adjacent to this angle. Make a sketch of the right triangle, look up the tangent of the given angle in the table, make an equation as shown below, and solve the equation.

EXAMPLE. In order to find the height of a flagpole, the distance AC and the angle at A were measured. If angle A is 65° and AC is 16 ft., how high is the flagpole?

$$\frac{BC}{AC} = \tan 65°$$



Substituting 16 for AC and 2.145 for tan 65° (found in the table), you have

$$\frac{BC}{16} = 2.145$$

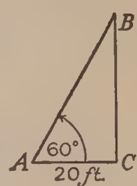
Multiplying each side by 16,

$$\begin{aligned} BC &= 16 \times 2.145 \\ &= 34.320 \end{aligned}$$

The flagpole is 34 ft. high (to the nearest foot).

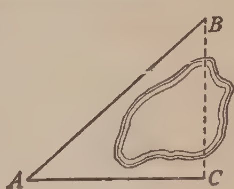
Exercises

1. Find the length of BC (to the nearest foot) in triangle ABC, when angle $A = 60°$ and $AC = 20$ ft. ($\angle C$ is 90°.)



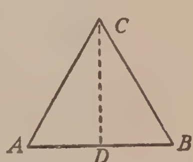
2. In triangle DEF, angle $F = 90°$, angle $D = 50°$, and $DF = 18$ ft. Find the length of EF to the nearest foot. (Sketch the triangle.)

3. In this figure BC represents the distance across a pond. Angle C is a right angle. What measurements would you make to find the length of BC?



4. Find the length of BC (to the nearest foot) in Ex. 3 when angle $A = 42°$ and $AC = 240$ ft.

5. In equilateral triangle ABC, CD makes a right angle with AB. Find the length of CD to the nearest unit when each side of triangle ABC is 28 units long. (AD is one half of AB. Each angle of an equilateral triangle is 60°.)



Angles of Elevation and Depression

Surveyors use the phrases *angle of elevation* and *angle of depression.* The following explanation will show you what they mean.

At the left of the figure is shown an angle of elevation. At the right is an angle of depression. In each case it is the angle between the horizontal line and the line of sight. When the point sighted is *above* the observer, this angle is called the ***angle of elevation.*** When the point sighted is *below* the observer, it is called the ***angle of depression.*** Remember that both angles of elevation and angles of depression are measured from the horizontal.

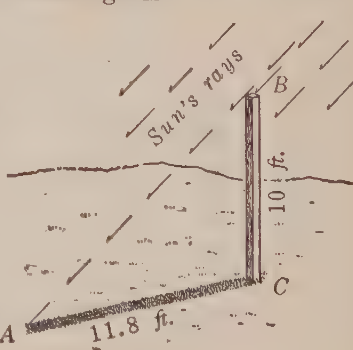
Exercises

1. The angle of elevation of the top of a tree from a point 80 ft. from its base (on level ground) is 60°. How high is the tree?

2. When an airplane is directly over point C, an observer at a point A, 460 yd. away, finds the angle of elevation to be 75°. How high is the plane?

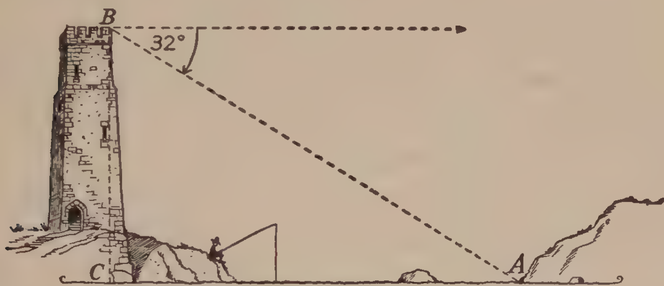
3. At noon a 10-ft. vertical pole cast a shadow 11.8 ft. long. What was the angle of elevation of the sun?

(Divide 10 by 11.8 to find the value of tan A. If you do not find this value in the table, choose the value in the table that is nearest to it.



4. What is the angle of elevation of the sun, when a pole 6 ft. long held vertically casts a shadow 4 ft. long?

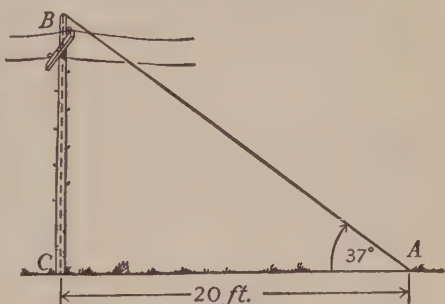
5. A tower 120 ft. high is built on the bank of a river. The angle of depression of a point A on the opposite bank is 32°. How wide is the river? (Note that the angle at B in the triangle ABC is $90° - 32° = 58°$.)



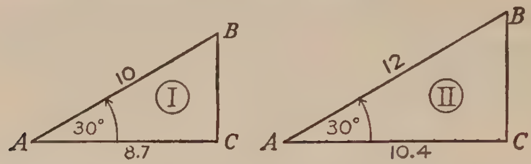
6. An aviator observes the angle of depression of a camp to be 38°. He is 5000 ft. above the ground. How far from camp is the point on the ground directly under the plane?

A Problem Requiring a New Ratio

The picture shows a telephone pole that needs bracing. Mr. Coleman, the repairman, wishes to find how long a wire he will need to reach from the stake A to the top of the telephone pole B. He has measured the line AC on the ground; it is 20 feet. He has measured the angle at A; it is 37°; the angle at C is 90°. He cannot use the tangent ratio, for the tangent does not have anything to do with the line AB. He uses another ratio, called the ***cosine of an angle.*** For any given angle in a right triangle the cosine is the ratio of the *side adjacent* to the *hypotenuse.*



(1) In triangle I, AC is about 8.7, $AB = 10$. What is the value, expressed decimally, of the ratio $\frac{AC}{AB}$?



The side opposite the right angle in a right triangle is called the ***hypotenuse*** of the triangle. AB is the hypotenuse.

(2) In triangle I, what is the ratio of the side adjacent to the 30° angle to the hypotenuse?

(3) In triangle II, AC is about 10.4, $AB = 12$. What is the value of the ratio $\frac{AC}{AB}$?

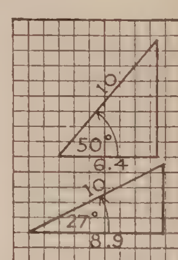
In right triangles, as these exercises illustrate, the *side adjacent* to a given acute angle divided by the *hypotenuse* is always the same ratio regardless of the size of the triangle. This ratio is called the ***cosine*** of the angle. Cosine A is usually written ***cos A***.

(4) On graph paper draw a right triangle in which $\angle C$ is 90°, $BC = 8$ units, $AC = 6$ units. You will find that AB is 10 units. Referring to this figure, answer these questions: How long is the hypotenuse? the side adjacent to $\angle B$? the side adjacent to $\angle A$? the side opposite $\angle B$? the side opposite $\angle A$?

(5) Referring to the triangle of Ex. (4), tell which of the following are correct:

(*a*) $\tan A = \frac{4}{3}$ (*c*) $\tan B = \frac{3}{4}$

(*b*) $\cos A = \frac{5}{4}$ (*d*) $\cos B = \frac{4}{5}$



(6) Using the figures at the right, answer this question: Is $\cos 50°$ greater or less than $\cos 27°$?

As the angle changes the cosine changes. The cosine decreases as the angle increases.

For any given acute angle in a right triangle the ratio of *the side adjacent to the angle to the hypotenuse* is the same for all right triangles.

(7) Can the side adjacent ever be equal to the hypotenuse? Can the cosine ever be equal to or greater than 1?

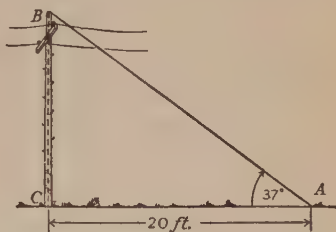
Using Cosines

Just as we used a table to find tangents, we can use a table to find cosines.

(1) Using the table on page 412, find cos 20°, cos 37°, and cos 74°.

(2) Using the table on page 412, find the angles whose cosines are as follows (give the angles to the nearest degree): .966, .891, .763, .547.

Here is the solution of Mr. Coleman's problem: (See page 417.)



$$\frac{AC}{AB} = \cos 37°$$

Substituting 20 for AC and .799 for cos 37°, we have,

$$\frac{20}{AB} = .799 \quad \text{or} \quad 20 = .799\,AB$$

Dividing both sides by .799,

$$\frac{20}{.799} = AB$$

Dividing, $25.03 = AB$

Hence a wire a little longer than 25 ft. will be a satisfactory brace for the pole.

Exercises

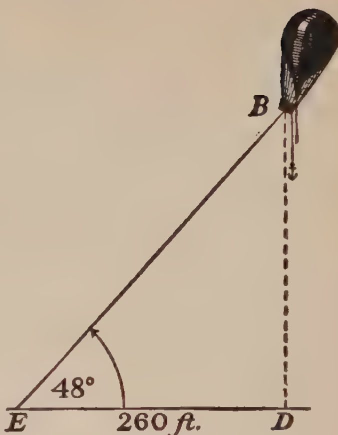
1. Using graph paper and a protractor, draw a right triangle having an acute angle of 40° and another right triangle having an acute angle of 70°. Measure the length of the side adjacent to the given angle and the hypotenuse in each triangle. Find the value of cos 40° and cos 70° by dividing the length of the side adjacent to the angle by the length of the hypotenuse. Check by referring to the table. (To find the length of the hypotenuse, transfer it to a horizontal or vertical line by means of a pair of compasses.)

2. Using the table, find the cosine of 35°; of 60°; of 75°; of 8°.

3. Using the table, find the angle whose cosine is .985; whose cosine is .819.

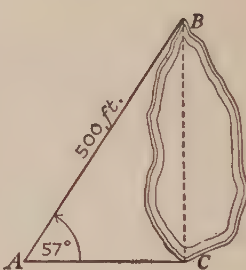
4. As the angle decreases, its cosine _?_.

5. A balloon B (see figure) is anchored to the ground at a point E by a wire making an angle of 48° with the ground. The point D on the ground directly under the balloon is 260 ft. from E. How long is the wire, assuming it to be straight?



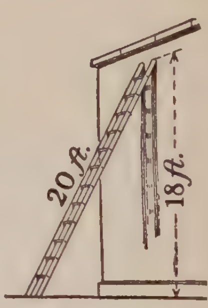
6. How far above the ground is the balloon shown in this picture? (Use the tangent ratio.)

7. The boys of Manhattan Troop 12 wanted to know the distance BC across a pond. They laid off the line CA at right angles to BC. They extended the line CA until they came to a point A from which they could measure AB. They measured AB and the angle A. If AB is 500 ft. and angle A is 57°, show how you would compute the length of BC. Find BC.



HINT. How large is angle B?

8. Note that the ladder in this picture is 20 ft. long and reaches the wall at a point 18 ft. from the ground. What angle does the ladder make with the house?



SUGGESTION. Cos $x = \frac{18}{20}$, or $\cos x = .900$; from the table, x is approximately _?_.

In this case you (1) compute a ratio and (2) turn to the table to find the size of an angle.

9. An airplane pilot glides to the ground from a height of 1300 ft. with an *angle of glide* (angle with the horizontal) of 10°. What is his *gliding distance* (horizontal distance traveled) before landing? (Should you use the tangent or the cosine?)

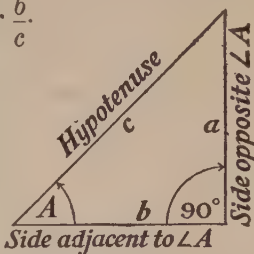
The Sine Ratio

The *tangent* of angle A in the right triangle below is

$$\frac{\textit{side opposite } \angle A}{\textit{side adjacent to } \angle A}, \text{ or } \frac{a}{b}.$$

The *cosine* of angle A in a right triangle with acute angles A and B is

$$\frac{\textit{side adjacent to } \angle A}{\textit{hypotenuse}}, \text{ or } \frac{b}{c}.$$



There is another important ratio, the $\frac{\textit{side opposite}}{\textit{hypotenuse}}$, called the ***sine*** ratio. The *sine* of $\angle A$ in a right triangle having the right angle at C is $\frac{a}{c}$. The sine of $\angle A$ is written ***sin A***.

These three ratios are called ***trigonometric ratios.***

Complete from the table on page 412:

(1) The sine of a 30° angle is __?__; this means that the side opposite the 30° angle in a right triangle divided by the hypotenuse is __?__.

(2) The sine of a 45° angle is __?__. What does this mean?

Suppose that you wanted to find the distance BC in the figure as shown here, and knew that $AB = 140$ ft. and $\angle A = 34°$. Your solution would look like this:

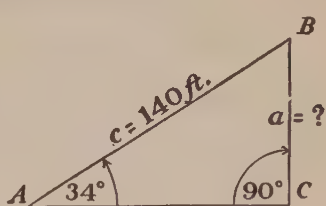
$$\frac{a}{140} = \sin 34°$$

From the table of sines, we get

$$\frac{a}{140} = .559$$

Multiplying each side by 140,

$$a = 78.26 \text{ ft.}$$



As the angle changes, the sine changes.

The sine increases as the angle increases.

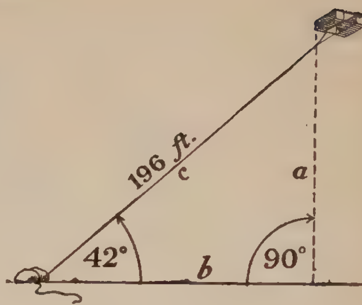
For any given acute angle in a right triangle, the ratio of *the side opposite the angle to the hypotenuse* is the same for all right triangles.

Exercises

1. John's kite string is 196 ft. long and makes an angle of 42° with the horizontal. How high is the kite above the ground? (Assume that the string is straight.) Which of these equations should you use:

$$\tan 42° = \frac{a}{b}, \text{ or } \cos 42° = \frac{b}{c},$$

or $\sin 42° = \frac{a}{c}$?

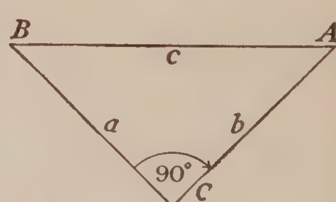


2. The railroad which runs to the summit of Pikes Peak makes, at the steepest place, an angle of 27° with the horizontal. How many feet would you rise in walking 109 ft. up the railroad track?

3. Refer to the figure below and complete:

(1) $\tan A =$?	(3) $\cos A =$?	(5) $\cos B =$?
(2) $\tan B =$?	(4) sine $A =$?	(6) sine $B =$?

4. These examples all refer to *right triangles.* Make a drawing for each example, using the same plan of lettering the sides and angles that we have been using.



(*a*) Find c when $a = 26$ and angle $A = 54°$.

(*b*) Find c when $a = 8.4$ and angle $A = 62°$.

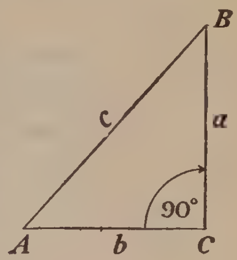
(*c*) Find a when $c = 65$ and angle $A = 45°$.

(*d*) Find b when $c = 100$ and angle $A = 35°$.

(*e*) Find b when $c = 42$ and angle $B = 30°$.

Using All Three Ratios

In solving these problems you should first make a drawing showing the parts that you know and the parts you wish to find. Then decide which ratio you should use to find the unknown parts.



1. Refer to the figure given here and complete: If I know side b and angle A, I can find side a by using the ? ratio. If I know side b and angle A, I can find the hypotenuse c by using the ? ratio. If I know side a and angle A, I can find side b by using the ? ratio; I can find the hypotenuse c by using the ? ratio.

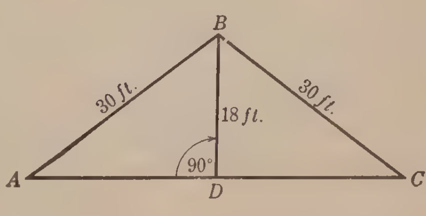
2. Complete: If I know angle B and side a, I can find side b by using the ? ratio, or I can find the hypotenuse c by using the ? ratio.

3. To find the distance across a pond between two points, M and N, Jack measured off 600 ft. on a line MR perpendicular to MN. He then found angle MRN to be 35°. Find the distance.

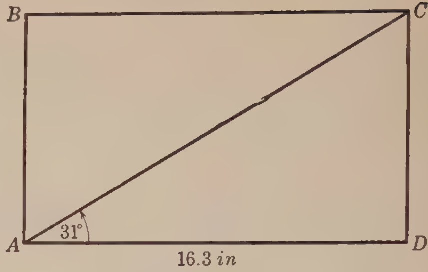
4. A balloon is anchored to the ground by a wire 580 ft. long. The wind blows the balloon so that the angle of elevation between the wire and a horizontal line is 25°. How high is the balloon above the ground? Is there any other distance that you could find from the facts given in the problem?

5. Some Girl Scouts measured the height of a mound. They stretched a string from a point, A, at the bottom of the mound to the top, T, finding AT to be 72.5 ft. The angle of elevation of the top from point A is 35°. How high is the mound?

6. If AB and BC are each 30 ft. and the height BD is 18 ft., how many degrees are there in angle A?



7. In the rectangle *ABCD*, the angle between *AD* and *AC* is 31° and *AD* = 16.3 in. How long is *AC*?



8. Find the angle of elevation of the sun to the nearest degree when a steeple 200 ft. high casts a horizontal shadow 80 ft. long.

9. From the top of a cliff 3500 ft. above a lake, the angle of depression of the nearest shore is 18°. Find the distance through the air from the top of the cliff to the edge of the lake.

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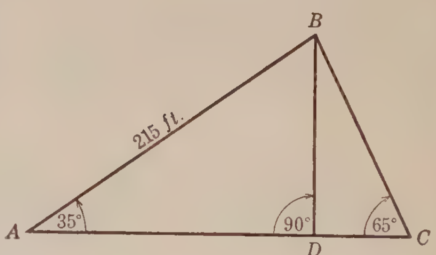
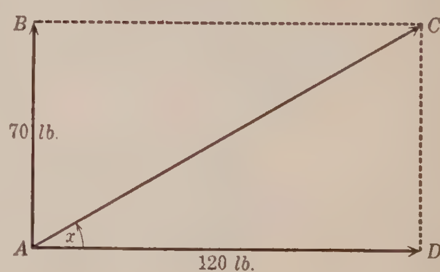
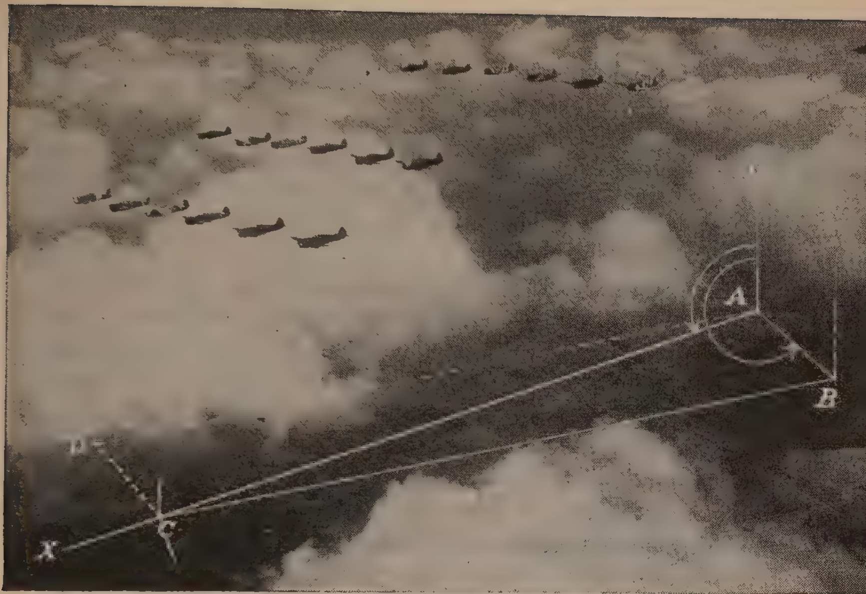
10. *AB* = 215 ft., angle *A* = 35°, angle *C* = 65°, and the angles at *D* are right angles. How long are *BD* and *BC*?



11. Two forces are acting on an object at *A*. One is pulling north with a magnitude of 70 lb. The other pulls east with a magnitude of 120 lb. Find the magnitude and direction of a single force equivalent to these two forces. (Let *AB* represent 70 lb. north and let *AD* represent 120 lb. east. Then *AC*, the diagonal of the rectangle, represents the magnitude of the required force, and the angle marked x gives its direction.)

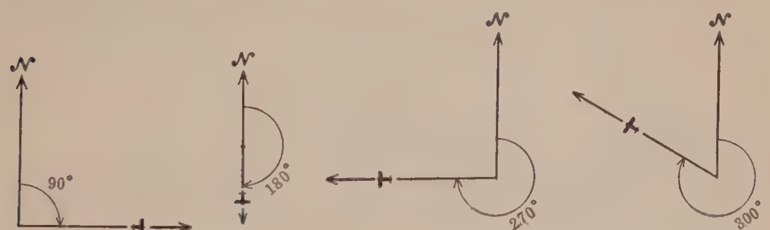




Airplane Problems

Direction, ground speed, air speed, course, and *track* are common words among those who fly planes. It is well to know the meaning of these terms and to gain some idea of how mathematics is used in determining the position and course of planes in the air.

Directions in aviation are given in degrees clockwise from the north from 0° to 360°. North is 0°; east is 90°; south is 180°; west is 270°; and northwest is 315°. The plane in the fourth figure below is flying at 300°.



(1) Using a protractor, make a diagram of a plane flying at 45°; at 170°; at 185°; and at 315°.

You are all familiar with the idea of ground speed. When an automobile travels at a constant rate of 35 miles an hour, it will cover 35 miles of ground in one hour. With autos the only speed we are interested in is ground speed, but with planes we have to consider air speed as well. *Air speed* is the speed of a plane relative to the surrounding air; *ground speed* is the speed relative to the ground below.

The air speed and the ground speed are not the same except in perfectly calm air. In an extreme case, it would be quite possible for a plane to have an air speed of 100 miles an hour and a zero ground speed. This would be true if the plane were traveling through the air at 100 miles an hour and a 100-mile-an-hour gale were blowing directly against it.

The direction in which an airplane is headed is called its *course.* Unless the air is calm, or the wind is blowing directly behind or in front, the plane will not follow this course but will drift in another path. The actual path is known as the *track.* *Air speed* is measured with the *course,* while *ground speed* is measured with the *track.*

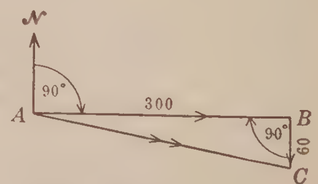
(2) If an airplane flies due east with an air speed of 300 miles an hour and the wind is blowing from the west with a speed of 40 miles an hour, what is the ground speed?

(3) If an airplane flies due east with an air speed of 300 miles an hour and the wind is blowing from the east at 40 miles an hour, what is the ground speed?

(4) In Exs. (2) and (3), is the track the same as the course or do they differ?

(5) An airplane is headed due east (90°) with an air speed of 300 miles an hour. At the same time a wind is blowing from the north at 60 miles an hour. Find the ground speed and the track.

SOLUTION. Let AB, drawn to the right (east), represent 300 and BC, drawn down (from the north), represent 60. Then the length of AC and its direction will represent the ground speed and the track.



$$\tan A = \frac{60}{300} = .200$$

$$A = 11° \text{ (to the nearest degree)}$$

Hence $\angle NAC = 101°$

ANSWER. The track is 101°.

To find the ground speed, use the cosine ratio. While it is theoretically correct to use the sine ratio, with small angles the sine (given to three places) may result in inaccuracies that give an absurd answer.

$$\frac{AB}{AC} = \frac{300}{AC} = \cos 11° = .982$$

$$AC = \frac{300}{.982} = 306 \text{ (to the nearest whole number)}$$

ANSWER. The ground speed is 306 miles an hour.

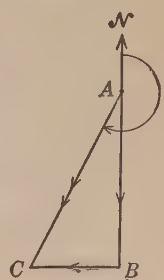
We are restricted as to the type of problem we can introduce here because you do not know how to solve oblique triangles as yet. The one type of problem solved above, however, will give you an idea of the elements involved.

Exercises

1. An airplane is headed due east with an air speed of 120 miles an hour while a 50-mile-an-hour wind is blowing from the south. Find the ground speed and the track.

2. A plane has an air speed of 150 miles an hour and is heading 180°. A 30-mile-an-hour wind is blowing from 90°. Find the track and the ground speed.

(To get the track, first find $\angle BAC$, then add 180° to get the angle shown by the curved arrow.)



(Ex. 2)

3. Given air speed, course, wind speed, and direction of wind as follows, find the track and the ground speed:

AIR SPEED	COURSE	WIND SPEED	WIND FROM
100 m.p.h.	180°	30 m.p.h.	west
180 m.p.h.	90°	50 m.p.h.	north
250 m.p.h.	270°	60 m.p.h.	south
300 m.p.h.	0°	45 m.p.h.	east

4. Find the ground speed of a plane with an air speed of 100 m.p.h., climbing at an angle of 12°.

5. If the ground speed of a plane just taking off is 100 m.p.h. and the angle of climb is 8°, what is the air speed?

6. If the track of a plane has been constantly 115° for 120 miles, how far south and how far east is it from its starting point?

7. If the track of a plane has been constantly 285° for 240 miles, how far north and how far west is it from its starting point?

---

8. A plane flies at 46° for 100 miles and then changes to 98°. If it continues on this second track for 100 miles, how far and in what direction is it from the starting point?

Chapter Summary

Numerical trigonometry is sometimes referred to as *indirect measurement.* It makes it possible to measure distances indirectly that it would be impossible or impractical to measure directly, by representing them as sides of *right triangles* and making use of the relation of the sides and angles of such triangles to estimate the distances.

Three important ratios in right triangles are used in indirect measurement:

$$\text{sine of acute } \angle A = \sin A = \frac{\text{side opposite } \angle A}{\text{hypotenuse}}$$

$$\text{cosine of acute } \angle A = \cos A = \frac{\text{side adjacent to } \angle A}{\text{hypotenuse}}$$

$$\text{tangent of acute } \angle A = \tan A = \frac{\text{side opposite } \angle A}{\text{side adjacent to } \angle A}$$

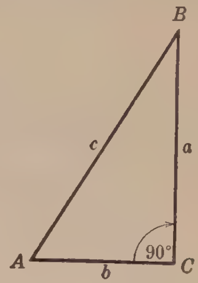
These — sine, cosine, and tangent — are called *trigonometric ratios.* For an acute angle of any given size these ratios are constant, no matter how large or how small the right triangle may be.

In a right triangle, when one side and an acute angle, or two sides, are known, you can find the unknown sides or angles by using these trigonometric ratios. These facts make numerical trigonometry a very useful instrument in many fields of world endeavor.

Chapter Review

1. Refer to the figure at the right and choose the right ratios for the following:

(1) sine A = ? (4) tan B = ?
(2) sine B = ? (5) tan A = ?
(3) cos A = ? (6) cos B = ?



2. You have learned that $\frac{a}{b} = \tan A$. What do you do to each side of this equation to get the equation $a = b \tan A$?

3. As an angle increases, its tangent ?.

4. As an angle increases, its cosine ?.

5. As an angle increases, its sine ?.

6. Using the tables, find the values of the following:

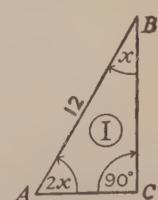
(a) cos 48° (d) tan 72°
(b) tan 37° (e) cos 84°
(c) sin 76° (f) sin 28°

7. Using the tables, find the angles which have the following trigonometric ratios:

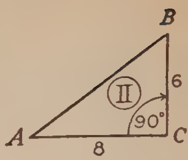
(a) tan A = 2.05 (d) tan A = .344
(b) cos A = .921 (e) sin A = .545
(c) sin A = .940 (f) cos A = .358

8. Using the tables, find to the nearest degree the angles which have the following trigonometric ratios:

(a) sin A = .351 (d) cos A = .505
(b) tan A = 1.894 (e) sin A = .710
(c) cos A = .973 (f) tan A = .329

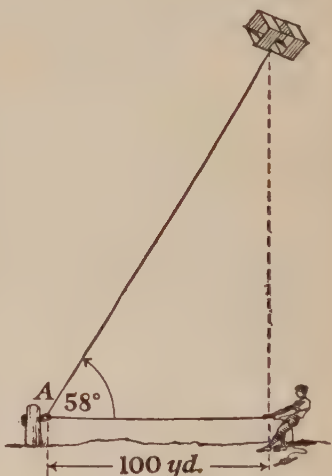


9. Using the data as given on triangle I, find the length of AC, the length of BC, and the number of degrees in angle B.



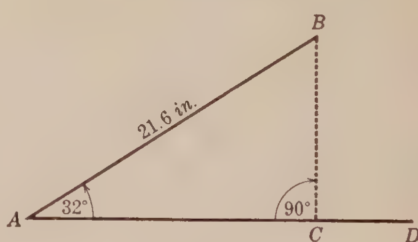
10. Using the data as given on triangle II, find to the nearest degree the size of angle A and the number of degrees in angle B.

11. John, who is 60 in. high, casts a shadow 20 in. long. Find the angle of elevation of the sun.



12. When the angle of elevation of the sun is 40°, what is the length of the shadow cast on level ground by a tree 80 ft. high?

13. This picture shows that the kite string makes an angle of 58° with the ground line. It also shows that Tom, who is standing directly under the kite, is 100 yd. from A. Find the height of the kite to the nearest yard.



14. AB is 21.6 in. long and makes an angle of 32° with AD. Find the projection of AB on AD. (The projection of AB on AD is AC.)

15. An airplane is headed due south with an air speed of 180 miles an hour. At the same time the wind is blowing 35 miles an hour from the west. Find the track and the ground speed.

Chapter XVIII

SQUARE ROOTS AND QUADRATIC EQUATIONS

A ***square root*** of a number is a number which multiplied by itself will give the number. It is one of two *equal* factors of a number. A ***quadratic equation*** is an equation of the second degree — one that contains the square of the unknown. Since one of the most important uses of square root is in the solution of quadratic equations, we shall study the two subjects together.

Positive and Negative Square Roots

Five is the square root of 25 because 5 multiplied by itself gives 25; that is, $5 \times 5 = 25$. The square root of $36x^6$ is $6x^3$ because $6x^3$ multiplied by itself gives $36x^6$, or $6x^3 \times 6x^3 = 36x^6$. It may not have occurred to you, however, that every positive number has two square roots.

Since $(+5)(+5) = 25$ and $(-5)(-5) = 25$, we see that the square root of 25 is either $+5$ or -5. The two square roots of 25 may be indicated by writing the double sign $\pm$ before the 5. Thus, ± 5 means "plus or minus five."

(1) Give the two square roots of each of the following:

(*a*) 36	(*c*) x^2	(*e*) $16a^2$	(*g*) $64y^8$
(*b*) 100	(*d*) a^4	(*f*) $49x^6$	(*h*) $.01y^2$

(2) If we know that $n^2 = 16$, we know that n equals either _?_ or _?_. We say that 4 or -4 is the _?_ of 16, or that 16 is the _?_ of 4 or -4.

The positive square root of a number is called its ***principal square root.*** When we speak of *the* square root of a number, we refer to the principal square root. Mathematicians have agreed that the radical sign, $\sqrt{\ }$, shall mean only the positive square root. Thus $\sqrt{25} = 5$, not ± 5, and $-\sqrt{25} = -5$. If you wish to indicate both roots, you must write $\pm\sqrt{25}$.

(3) Give the value of each of the following:

(*a*) $\sqrt{4}$	(*c*) $-\sqrt{36}$	(*e*) $\sqrt{x^2}$	(*g*) $-\sqrt{36x^{10}}$
(*b*) $\sqrt{144}$	(*d*) $-\sqrt{49}$	(*f*) $-\sqrt{a^4}$	(*h*) $\pm\sqrt{81a^4}$

Square Roots of Monomials

To find the square root of a monomial, take the square root of the numerical coefficient and divide each exponent by two.

Thus, $\sqrt{25a^4b^2c^6} = 5a^2bc^3$. CHECK. $(5a^2bc^3)(5a^2bc^3) = 25a^4b^2c^6$.

The principal square root of $49a^8b^6$ is $7a^4b^3$; its other square root is $-7a^4b^3$.

To find the square root of a fraction, take the square root of both the numerator and the denominator.

Thus, both square roots of $\frac{4}{9}$ are $\frac{2}{3}$ and $-\frac{2}{3}$. Note, however, that $\sqrt{\frac{4}{9}}$ is $\frac{2}{3}$ and $-\sqrt{\frac{4}{9}} = -\frac{2}{3}$.

Exercises

Find the principal square root of each of the following:

1. $64x^4$	**7.** $.81y^2$	**13.** $100x^2y^6$
2. $36x^2$	**8.** $.16t^4$	**14.** $169a^4b^2c^8$
3. $100y^2$	**9.** $16x^{16}$	**15.** $121x^6y^{12}$
4. $25x^2$	**10.** $36a^{36}$	**16.** $.25a^6b^{10}$
5. $64x^2y^2$	**11.** $81a^4b^6$	**17.** $1.44a^6b^8$
6. $9a^4b^2$	**12.** $100x^8y^2$	**18.** $\frac{4}{25}x^8y^6$

Find the value of each of the following:

19. $\sqrt{16x^4}$	**23.** $\sqrt{\frac{4x^8}{25y^6}}$	**26.** $\sqrt{(a+b)^2}$
20. $-\sqrt{81a^4b^2}$	**24.** $\pm\sqrt{\frac{100x^2}{49y^{10}}}$	**27.** $\sqrt{4(x-y)^4}$
21. $-\sqrt{4x^8}$	**25.** $-\sqrt{\frac{25}{81}x^2y^6}$	**28.** $\sqrt{\frac{196a^4b^2}{81c^{10}}}$
22. $\pm\sqrt{16y^8}$		

Square Roots by Computation

Find the square root of 1796.34 to the nearest tenth.

SOLUTION. Begin at the decimal point and mark off the digits to the right and left in groups of two. (The first group at the left will contain only one digit if there is an odd number of digits to the left of the decimal point. The number of digits to the right of the decimal point may always be made even by annexing zeros.)

In the example, the largest perfect square less than 17, the first group of numbers, is 16. Hence the first digit in the square root is 4.

Square the 4, getting 16. Subtract 16 from 17, write down the result, 1, and bring down the next number group, 96; this gives 196. To find the first trial divisor, annex a zero to the partial answer you have found and double it: $2 \times 40 = 80$.

Dividing 196 by 80, you find that the next digit in your answer is probably 2. The complete divisor is $80 + 2$, or 82.

```
          4  2. 3  8
        17 96.34 00
        16
  82 |  196
        164
 843 |  3234
        2529
8468 |  70500
        67744
         2756
```

$\sqrt{1796.34} = 42.4$ correct to the nearest tenth.

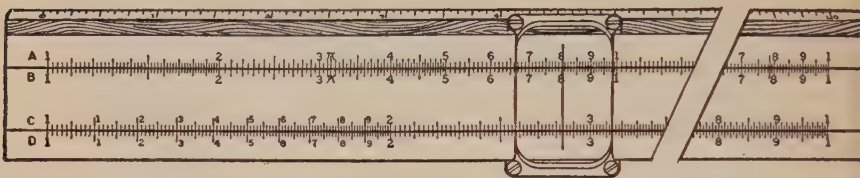
Multiply 82 by 2, the number you added to 80. $82 \times 2 = 164$. (If, when you multiplied, you had obtained a number greater than 196, you would have known that 82 was too large. You would then have tried 81 and multiplied by 1.)

Subtract 164 from 196, giving 32; then bring down the next group of digits. This gives 3234. From this point on the method repeats itself.

Annex a zero to the 42 in your answer and double it: $2 \times 420 = 840$. Divide 3234 by 840. The next digit in the answer is probably 3. The next complete divisor is therefore 843. Multiply $3 \times 843 = 2529$. Subtract 2529 from 3234. The remainder is 705.

Carry the process one step further to find whether the next digit in the answer is 5 or more. In this case it is obviously more than 5. Hence the square root of 1796.34 to the nearest tenth is 42.4.

This method of finding square roots is derived from the statement $(a + b)^2 = a^2 + 2\,ab + b^2$. The given number is $a^2 + 2\,ab + b^2$ and you are asked to find $a + b$. In the example solved, the first number you found — namely, 4 — is a. You then subtracted a^2 and had $2\,ab + b^2$ left. This can be written as $b(2\,a + b)$. This expression shows why you double the answer (getting $2a$), add the quotient when you divide (getting $(2a + b)$), and then multiply by the quotient (getting $b(2a + b)$).



The slide rule is an instrument, based on logarithms, which saves time and labor in many calculations of the businessman, engineer, mechanic, accountant, and statistician.

Exercises

Compute the square root of each of the following. (If the numbers are not squares, give the root correct to the nearest hundredth.)

1. 529	**10.** 834	**19.** 700
2. 1024	**11.** 67.3	**20.** 70
3. 1156	**12.** 2	**21.** 7
4. 9216	**13.** 3	**22.** .7
5. 156	**14.** 5	**23.** .07
6. 6.8342	**15.** .0653	**24.** 9.7
7. 23.644	**16.** 30	**25.** 104
8. 39.98	**17.** 300	**26.** 50.04
9. 725	**18.** 12.3456	**27.** 64.08

28. Find the square root of .25.

29. Find correct to hundredths the length of a side of a square whose area is 80 sq. in.

30. Find correct to tenths the number of rods in the perimeter of a square field whose area is 5 acres. (1 A. = 160 sq. rd.)

31. A farmer wanted to mark off a square piece of ground that would contain one acre. How long should he make each side of the square?

Some Square Roots You Should Remember

Students of mathematics and the sciences need to use $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ so frequently that they usually memorize the approximate results, thus:

$$\sqrt{2} = 1.414 \cdots; \quad \sqrt{3} = 1.732 \cdots; \quad \sqrt{5} = 2.236 \cdots$$

Answer these questions by using the approximate values given above. Does $\sqrt{2} + \sqrt{3}$ equal $\sqrt{5}$? Does $\sqrt{2} \times \sqrt{3}$ equal $\sqrt{5}$? Does $\sqrt{3} - \sqrt{2} = \sqrt{1}$?

Using a Table of Squares and Square Roots

In practical computations it is convenient to use a table of squares and square roots. Such tables may be had at small cost. It may be interesting for you to examine the more extensive ones used by statisticians. A very simple table of squares and square roots is printed on page 437 for your convenience.

EXAMPLE 1. Find the square of 47. Locate 47 in the column headed "No." Then in the same row in the column headed "Square" you will find 2209, which is 47^2.

EXAMPLE 2. Find the square root of 47. Locate 47 as before. Then in the same row in the column headed "Square Roots" you will find 6.856. This is $\sqrt{47}$ correct to the nearest thousandth.

Note also that any number in a column headed "Squares" has its square root in the column headed "No." Thus, $\sqrt{14{,}641}$ is 121.

Using the table, discover whether $\sqrt{12}$ is equal to $4\sqrt{3}$ or $2\sqrt{3}$.

Exercises

Using the table, write the squares of the following numbers:

1. 13
2. 32
3. 51
4. 82
5. 101
6. 98
7. 145
8. 63
9. 132

Using the table, find the principal square root of each of the following numbers correct to hundredths. (You will need to round off the numbers in the table.)

10. 3
11. 5
12. 10
13. 20
14. 32
15. 44
16. 63
17. 70
18. 82
19. 88
20. 95
21. 142
22. 136
23. 150
24. 18,496
25. 8,649
26. 20,164
27. 11,664

28. Find correct to hundredths the square root of 68.5. (Find a number halfway between $\sqrt{68}$ and $\sqrt{69}$.)

Find the square root of each of the following numbers correct to hundredths:

29. 132.5 **30.** 98.5 **31.** 69.5 **32.** 40.5

33. Compare the square roots of two numbers whose ratio is 2 to 1, say 20 and 10 or 50 and 25. Are the square roots in the ratio 2 to 1?

34. Compare the square roots of two numbers whose ratio is 4 to 1, say 20 and 5 or 48 and 12. What is the ratio of the square roots?

Computing the Square Root of a Fraction

A fraction can be changed in form so that it is easier to get its square root. The denominator should be made a square by multiplying the numerator and the denominator of the fraction by a suitable number.

(1) Find the square root of $\frac{3}{5}$. If we should take the square root of both numerator and denominator as the fraction stands, we should have $\frac{1.732}{2.236} = .77$ (This requires long division.)

Instead, change the form of $\frac{3}{5}$ as follows: $\frac{3}{5} = \frac{3 \times 5}{5 \times 5} = \frac{15}{25}$.

Here the denominator is a square. $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \frac{\sqrt{15}}{\sqrt{25}} = \frac{3.87}{5}$ $= .77$. (The division here is much simpler in the second example.)

Table of Squares and Square Roots

Num.	Square	Sq. Root	Num.	Square	Sq. Root	Num.	Square	Sq. Root
1	1	1.000	51	26 01	7.141	101	1 02 01	10.050
2	4	1.414	52	27 04	7.211	102	1 04 04	10.100
3	9	1.732	53	28 09	7.280	103	1 06 09	10.149
4	16	2.000	54	29 16	7.348	104	1 08 16	10.198
5	25	2.236	55	30 25	7.416	105	1 10 25	10.247
6	36	2.449	56	31 36	7.483	106	1 12 36	10.296
7	49	2.646	57	32 49	7.550	107	1 14 49	10.344
8	64	2.828	58	33 64	7.616	108	1 16 64	10.392
9	81	3.000	59	34 81	7.681	109	1 18 81	10.440
10	1 00	3.162	60	36 00	7.746	110	1 21 00	10.488
11	1 21	3.317	61	37 21	7.810	111	1 23 21	10.536
12	1 44	3.464	62	38 44	7.874	112	1 25 44	10.583
13	1 69	3.606	63	39 69	7.937	113	1 27 69	10.630
14	1 96	3.742	64	40 96	8.000	114	1 29 96	10.677
15	2 25	3.873	65	42 25	8.062	115	1 32 25	10.724
16	2 56	4.000	66	43 56	8.124	116	1 34 56	10.770
17	2 89	4.123	67	44 89	8.185	117	1 36 89	10.817
18	3 24	4.243	68	46 24	8.246	118	1 39 24	10.863
19	3 61	4.359	69	47 61	8.307	119	1 41 61	10.909
20	4 00	4.472	70	49 00	8.367	120	1 44 00	10.954
21	4 41	4.583	71	50 41	8.426	121	1 46 41	11.000
22	4 84	4.690	72	51 84	8.484	122	1 48 84	11.045
23	5 29	4.796	73	53 29	8.544	123	1 51 29	11.091
24	5 76	4.899	74	54 76	8.602	124	1 53 76	11.136
25	6 25	5.000	75	56 25	8.660	125	1 56 25	11.180
26	6 76	5.099	76	57 76	8.718	126	1 58 76	11.225
27	7 29	5.196	77	59 29	8.775	127	1 61 29	11.269
28	7 84	5.292	78	60 84	8.832	128	1 63 84	11.314
29	8 41	5.385	79	62 41	8.888	129	1 66 41	11.358
30	9 00	5.477	80	64 00	8.944	130	1 69 00	11.402
31	9 61	5.568	81	65 61	9.000	131	1 71 61	11.446
32	10 24	5.657	82	67 24	9.055	132	1 74 24	11.489
33	10 89	5.745	83	68 89	9.110	133	1 76 89	11.533
34	11 56	5.831	84	70 56	9.165	134	1 79 56	11.576
35	12 25	5.916	85	72 25	9.220	135	1 82 25	11.619
36	12 96	6.000	86	73 96	9.274	136	1 84 96	11.662
37	13 69	6.083	87	75 69	9.327	137	1 87 69	11.705
38	14 44	6.164	88	77 44	9.381	138	1 90 44	11.747
39	15 21	6.245	89	79 21	9.434	139	1 93 21	11.790
40	16 00	6.325	90	81 00	9.487	140	1 96 00	11.832
41	16 81	6.403	91	82 81	9.539	141	1 98 81	11.874
42	17 64	6.481	92	84 64	9.592	142	2 01 64	11.916
43	18 49	6.557	93	86 49	9.644	143	2 04 49	11.958
44	19 36	6.633	94	88 36	9.695	144	2 07 36	12.000
45	20 25	6.708	95	90 25	9.747	145	2 10 25	12.042
46	21 16	6.782	96	92 16	9.798	146	2 13 16	12.083
47	22 09	6.856	97	94 09	9.849	147	2 16 09	12.124
48	23 04	6.928	98	96 04	9.899	148	2 19 04	12.166
49	24 01	7.000	99	98 01	9.950	149	2 22 01	12.207
50	25 00	7.071	100	1 00 00	10.000	150	2 25 00	12.247

To find the square root of a fraction whose denominator is not a square —

Multiply the numerator and the denominator of the fraction by a number which will make the denominator a perfect square. Then —

Find the approximate square root of the numerator, and divide this result by the square root of the denominator.

(2) Find the square root of $\frac{5}{8}$ correct to hundredths.

Multiply numerator and denominator by 2: $\frac{5}{8} = \frac{10}{16}$

$$\sqrt{\frac{5}{8}} = \sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{\sqrt{16}} = \frac{\sqrt{10}}{4}$$

Find $\sqrt{10}$ from the table. It is 3.162.

$$\frac{\sqrt{10}}{4} = \frac{3.162}{4} = .791 \text{ or } .79 \text{ (correct to hundredths).}$$

Exercises

Find the square root of these fractions, correct to hundredths:

1. $\frac{2}{3}$	**4.** $\frac{2}{9}$	**7.** $\frac{3}{8}$	**10.** $\frac{7}{8}$
2. $\frac{2}{5}$	**5.** $\frac{5}{6}$	**8.** $\frac{3}{10}$	**11.** $\frac{1}{8}$
3. $\frac{2}{7}$	**6.** $\frac{3}{7}$	**9.** $\frac{4}{7}$	**12.** $\frac{9}{4}$

Two New Axioms

Throughout your study of algebra you have been using axioms when you solved equations. The equation $3x = 15$ may be changed into the equation $x = 5$ by dividing both sides by 3 (using the division axiom); the equation $\frac{n}{4} = 6$ may be changed into the equation $n = 24$ by multiplying both sides by 4 (using the multiplication axiom).

Two new axioms are used in solving equations which contain roots and powers. These are:

Like roots of equal numbers are equal.
Like powers of equal numbers are equal.

Consider these equations:

(1) $x^2 = 25$

(2) $x = +5 \text{ or } x = -5$

Equation (2) comes from equation (1) by finding the square root of each member of the equation. This is an application of the first axiom above. Equation (1) shows equal numbers; equation (2) shows like roots (in this case *square* roots) of the equal numbers in (1).

On the other hand, the equation $\sqrt{x} = 5$ may be changed to the equation $x = 25$ by applying the second axiom above. We find the second power (square) of each member of the first equation to get the second equation. In checking, note that the square of $\sqrt{5}$ is 5.

EXAMPLE 1. Solve: $x^2 + 5 = 9$

Subtracting 5, $x^2 = 4$

Taking square root, $x = \pm 2$

EXAMPLE 2. Solve: $\frac{n^2 - 1}{3} = 4$

Multiplying by 3, $n^2 - 1 = 12$

Adding 1, $n^2 = 13$

Taking square root, $n = \pm\sqrt{13}$

$= \pm 3.61$ (correct to the nearest hundredth)

Exercises

1. What axiom is used in solving the equation $x^2 = 36$?
2. What axiom is used to solve the equation $\sqrt{x} = 4$?

Solve each of the equations, keeping in mind the axioms which make the solutions possible:

3. $x^2 = 9$
4. $n^2 = 16$
5. $a^2 = 8$
6. $x^2 = 30$
7. $4\,m^2 = 64$
8. $3\,x^2 = 27$
9. $n^2 + 1 = 17$
10. $a^2 - 3 = 33$
11. $2\,n^2 + 5 = 55$
12. $x^2 + 2 = 3$
13. $x^2 - 7 = 7$
14. $\frac{1}{2}x^2 = 50$
15. $\frac{1}{3}n^2 = 6$
16. $\sqrt{x} = 10$

17. $\sqrt{n} = 2$

18. $n^2 = \frac{2}{3}$

19. $8n^2 = 5$

20. $\frac{n^2}{2} + 5 = 7$

21. $\frac{x^2}{9} + 1 = 5$

22. $\frac{n^2}{5} + \frac{1}{2} = \frac{11}{2}$

23. $\frac{n^2}{2} + \frac{1}{3} = \frac{29}{6}$

24. $\frac{n^2}{4} - \frac{1}{2} = 4$

25. $a^2 - \frac{1}{4} = 3$

26. $\frac{x - 1}{3} = \frac{6}{x + 1}$

27. $\frac{a + 1}{2} = \frac{4}{a - 1}$

28. $\frac{3}{2} = \frac{6}{\sqrt{x}}$

Quadratic Equations

The problems you have solved in previous chapters have involved equations of the first degree. Below is a problem that leads to an equation of a different kind.

PROBLEM. Many artists believe that a rectangle is most pleasing to the eye when its length and width satisfy the following proportion: $\frac{w}{l} = \frac{l}{l + w}$. Assuming this principle to be true, what should be the dimensions of a rectangle whose perimeter is 8 inches?

If the perimeter is 8, then $l + w = 4$. Why?

If the length is represented by x, then the width is $4 - x$.

Substituting in the proportion, you have: $\frac{4 - x}{x} = \frac{x}{4}$.

Multiplying both sides of this equation by $4x$, you get $16 - 4x = x^2$.

This equation will be solved later.

Note that this is not an equation of the first degree, since it contains the second power of the unknown x. It is an *equation of the second degree* or a *quadratic equation.*

Any equation which by the use of the addition and subtraction axioms can be changed into the form $ax^2 + bx + c = 0$, where x is an unknown that may be denoted by any letter, a, b, and c are any constants, and a is not zero, is an equation of the second degree.

The equation $10x - x^2 = 28$ can be changed to $x^2 - 10x + 28 = 0$, and it is therefore a quadratic equation.

The equation $2x^2 - 3x + 5 = 3x + 2x^2$ is not a quadratic equation. When $2x^2$ is subtracted from both sides, the second power of the variable disappears; that is, the coefficient of x^2 is 0.

If b is 0 in the equation $ax^2 + bx + c = 0$, the equation becomes $ax^2 + c = 0$. Such an equation is called an ***incomplete quadratic equation***, since the term containing the first power of the unknown is missing.

Examples of incomplete quadratic equations are: $x^2 - 9 = 0$, $y^2 = 8$, $2x^2 + 3 = x^2 - 5$.

A quadratic equation which has the first power of the unknown as well as the second power is called a ***complete quadratic equation***. An example of a complete quadratic equation is $3x^2 + 2x = 5$.

Solving Incomplete Quadratic Equations

An incomplete quadratic equation is easy to solve algebraically, since it can always be changed into the form $x^2 = k$, where k is any constant. Then all you have to do is to take the square root of both sides of the equation. You have already solved several equations of this type which you found on page 439.

EXAMPLE 1. Solve $x^2 - 9 = 0$

Adding 9, $x^2 = 9$

Taking square root, $+x = \pm 3$ or

$-x = \pm 3$

This gives (1) $x = 3$, (2) $x = -3$, (3) $-x = 3$, (4) $-x = -3$. It would seem at first sight that this equation has four roots (numbers satisfying it). But if you multiply both sides of equations (3) and (4) by -1, you get $x = -3$ and

$x = 3$. These are the same roots as found in equations (2) and (1). It is therefore unnecessary to write $-x = \pm 3$ in the original solution.

The roots of this equation are $+3$ and -3. The solution of such equations must include both roots; that is, $x = +3$ and $x = -3$, or $x = \pm 3$.

Every quadratic equation has two roots.

EXAMPLE 2. Solve $\frac{n^2}{2} - \frac{n}{3} = \frac{9-n}{3}$

When this equation is cleared of fractions, it becomes an incomplete quadratic.

Multiplying by 6, $3n^2 - 2n = 18 - 2n$

Adding $2n$, $3n^2 = 18$

Dividing by 3, $n^2 = 6$

Taking square root, $n = \pm\sqrt{6}$

The two roots of this equation are $+\sqrt{6}$ and $-\sqrt{6}$.

You can find the approximate value of these roots from the table on page 437.

Exercises

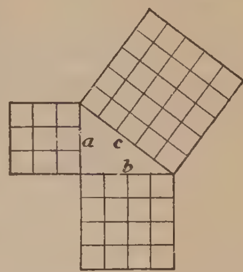
Solve the following equations. (Give approximate roots correct to hundredths.)

1. $x^2 = 100$
2. $n^2 = 81$
3. $a^2 - 36 = 0$
4. $c^2 - 49 = 0$
5. $3x^2 = 75$
6. $9x^2 = 36$
7. $n^2 = 7$
8. $y^2 = 8$
9. $y^2 = 20$
10. $3n^2 + 7 = 28$
11. $4a^2 - 9 = 11$
12. $3x^2 - 2 = x^2 + 6$
13. $x^2 - 7 = 5 - 3x^2$
14. $\frac{n^2}{3} - \frac{n}{2} = \frac{4-2n}{4}$
15. $\frac{n^2}{5} + \frac{n}{3} = \frac{n+3}{3}$
16. $\frac{n^2}{5} + \frac{n}{3} + \frac{1}{30} = \frac{2n+5}{6}$
17. $4x^2 = 3$
18. $9x^2 = 2$
19. $\frac{2n^2-1}{3} = \frac{1}{2}$
20. $\frac{x^2}{4} + 5 = \frac{x^2+2}{2}$

The Law of Pythagoras

The Law of Pythagoras is: *The square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides.*

How does the figure shown here illustrate this law? Stated algebraically, the Law of Pythagoras is $c^2 = a^2 + b^2$.



If two of the three numbers in this equation are known, the third number can be found.

EXAMPLE 1. If $a = 3$ and $b = 4$, how long is the hypotenuse c?

Use the relation $c^2 = a^2 + b^2$. Substitute 3 for a and 4 for b.

Then $c^2 = 9 + 16 = 25$
$c = \pm 5$

Since the answer -5 has no meaning in this problem, it is rejected. The hypotenuse is 5.

EXAMPLE 2. If $b = 7$ and $c = 11$, how long is a?

Use the relation $c^2 = a^2 + b^2$. Substitute 7 for b and 11 for c.

Then $121 = a^2 + 49$. $a^2 = 72$ and $a = \sqrt{72}$, or 8.49 (correct to hundredths.)

Exercises

If the base and altitude of a right triangle are as follows, how long is the hypotenuse in each case? (Give approximate answers correct to hundredths.)

1. 6 and 8
2. 5 and 12
3. 12 and 16
4. 24 and 10
5. 3 and 2
6. 4 and 7
7. 9 and 5
8. 8 and 7

If the hypotenuse and base of a right triangle are as follows, how long is the altitude in each case?

9. 10 and 6
10. 13 and 12
11. 5 and 4
12. 9 and 7
13. 15 and 10
14. 13 and 8

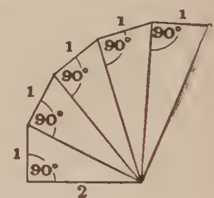
Which of these equations apply to a right triangle? (c *is the hypotenuse.*)

15. $c^2 - a^2 = b^2$ **17.** $b^2 - c^2 = a^2$ **19.** $b = \sqrt{c^2 + a^2}$

16. $a^2 = c^2 - b^2$ **18.** $c = \sqrt{a^2 + b^2}$ **20.** $c^2 = (a + b)^2$

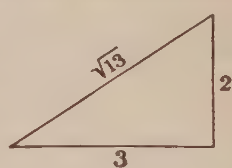
21. The hypotenuse of an isosceles right triangle is 2 units. How long are the sides? (Let each side be x and form an equation.)

22. The diagonal of a square is 2 units long. How long are its sides?



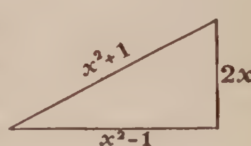
23. This figure shows a series of right triangles, each with an altitude of 1 unit. The hypotenuse of the first triangle becomes the base of the next triangle, and so on. Compute the length of each hypotenuse.

Do you see any short cut for finding the length of any particular line, such as the fifth one, without finding the length of the preceding ones?



24. Is the triangle shown here a right triangle? It is a right triangle if the square of the longest side equals the sum of the squares of the other two sides.

25. The three sides of a triangle are 4, 5, and $\sqrt{41}$. Is it a right triangle? Why? What is its area?



26. The lengths of the sides of a triangle are as shown in this figure. Is it a right triangle? Check your answer by using some arithmetical value for x, or by testing whether $(x^2 + 1)^2 = (x^2 - 1)^2 + (2\,x)^2$.

27. A baseball diamond is a square 90 feet on a side. How far is it from the home plate to second base? from first base to third base?

28. If a plane traveling at 150 m.p.h. goes 450 miles due north and then 600 miles due west, how long will it take it to fly home by the shortest route?

Solving Complete Quadratics by Formula

Every complete quadratic equation can be put in the form $ax^2 + bx + c = 0$ and can then be solved by means of a formula.

(1) The following equations are all in the form $ax^2 + bx + c = 0$. In each case state the value of a, b, and c. (In (*a*), $a = 1$, $b = -7$, and $c = 3$.)

(*a*) $x^2 - 7x + 3 = 0$

(*b*) $2x^2 + 5x - 4 = 0$

(*c*) $x^2 - 3x + 5 = 0$

(*d*) $5n^2 + n - 3 = 0$

(*e*) $3x^2 - x + 1 = 0$

(*f*) $b^2 + b + 1 = 0$

(2) Change the following equations to the form $ax^2 + bx + c = 0$:

(*a*) $x^2 + 5x = -3$

(*b*) $2x^2 - 3x = 7$

(*c*) $5n^2 = 3 + 2n$

(*d*) $3x + 2 = x^2$

(*e*) $7 = 2y^2 - 3y$

(*f*) $9 + 2x = 7x^2$

The formula for solving such equations is —

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that there are two values of x (two roots) as follows:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

You will learn how this formula is derived in a later course in algebra.

> **To solve a complete quadratic equation, put it in the form $ax^2 + bx + c = 0$, determine the values of a, b, and c, and then substitute these values in the formula.**

EXAMPLE 1. Solve $x^2 - 6x = -8$.

SOLUTION. Change to $x^2 - 6x + 8 = 0$, where $a = 1$, $b = -6$, and $c = 8$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting, $x = \dfrac{-(-6) \pm \sqrt{36 - 4(1)(8)}}{2}$

$$x = \frac{6 \pm \sqrt{36 - 32}}{2} = \frac{6 \pm \sqrt{4}}{2} = \frac{6 \pm 2}{2}$$

$$x = \frac{6 + 2}{2} = 4 \text{ or } x = \frac{6 - 2}{2} = 2$$

$$x = 4 \text{ or } 2$$

CHECK. When $x = 4$, $x^2 - 6x$ becomes
$4^2 - 6(4) = 16 - 24 = -8.$
When $x = 2$, $x^2 - 6x$ becomes
$2^2 - 6(2) = 4 - 12 = -8.$

EXAMPLE 2. Solve $3x^2 + 2x - 3 = 0.$

SOLUTION. $a = 3$, $b = 2$, $c = -3.$

Then, $x = \dfrac{-2 \pm \sqrt{4 - 4(3)(-3)}}{6}$

$$x = \frac{-2 \pm \sqrt{4 + 36}}{6} = \frac{-2 \pm \sqrt{40}}{6}$$

To get approximate values find $\sqrt{40}$ from the tables.

$$x = \frac{-2 \pm 6.325}{6}$$

$$x = \frac{-2 + 6.325}{6} = \frac{4.325}{6} = .72 \text{ (correct to hundredths)}$$

$$x = \frac{-2 - 6.325}{6} = \frac{-8.325}{6} = -1.39 \text{ (correct to hundredths)}$$

Some quadratic equations have roots like these:

$$x = \frac{-2 \pm \sqrt{-49}}{9}$$

Here you have the square root of a negative number which you must consider impossible at this stage of your knowledge of algebra.

Exercises

Solve the following quadratic equations:

1. $x^2 + 4x = 12$
2. $x^2 + 6x = 27$
3. $2x^2 - x - 3 = 0$
4. $2x^2 + x = 3$
5. $3x^2 = 2 - x$
6. $n^2 - 2n - 8 = 0$

7. $a^2 + 4a = 5$
8. $6a^2 + 13a + 6 = 0$
9. $n^2 + 6 = -5n$
10. $2y^2 = 5y - 3$
11. $x^2 = 3x$*
12. $p^2 - 4p = 0$
13. $y^2 + 3y = 4$
14. $y^2 - y = 2$
15. $10b^2 + 7b - 12 = 0$
16. $2a^2 + 11a + 5 = 0$
17. $b^2 = 10b$
18. $x^2 - 9x = -18$
19. $2x^2 = 7$
20. $4a^2 - 4a + 1 = 0$

Did you notice that in solving for a in Ex. 20, the quantity under the radical, $b^2 - 4ac$, was equal to zero and you could only get one value $(+\frac{1}{2})$ for the root? This means that the equation has two roots with the same value, $+\frac{1}{2}$. When $b^2 - 4ac = 0$, a quadratic equation of the form $ax^2 + bx + c = 0$ has two identical roots.

Also, did you observe that in Exs. 11, 12, and 17, where c was equal to zero, one of the roots was zero?

Solve the following equations:

21. $2x^2 - 7x + 3 = 0$
22. $3a^2 + 8a + 5 = 0$
23. $3n^2 - 2n = 1$
24. $3x^2 - x - 2 = 0$
25. $2a^2 - 5a - 12 = 0$
26. $2n^2 = n + 3$
27. $2x^2 - x - 3 = 0$
28. $3x^2 - 2x = 8$
29. $5n^2 + 2 = 11n$
30. $3b^2 + 5b + 2 = 0$
31. $2y^2 - 3y = 35$
32. $4n^2 + 4 = 17n$
33. $2a^2 + 3a = 20$
34. $1 = 6x^2 - x$
35. $2 - y = 3y^2$
36. $8y^2 - 2y = 3$
37. $2x^2 = 9x$
38. $8p^2 - 2p - 1 = 0$
39. $10y^2 - 41y = -4$
40. $6p^2 - 9p = 27$

Find the roots of the following equations correct to the nearest hundredth:

41. $x^2 - 2x = 4$
42. $x^2 + 2x = 1$
43. $x^2 + 4x = -2$
44. $x^2 + 6x + 3 = 0$

* In Ex. 11, $c = 0$.

45. $x^2 - x - 1 = 0$

46. $x^2 + 3x + 1 = 0$

47. $2x^2 - 4x - 3 = 0$

48. $2x^2 + x - 5 = 0$

49. $3x^2 - 2x = 1$

50. $2x^2 + 7x + 2 = 0$

51. $6x^2 - 3x - 4 = 0$

52. $3x^2 + 10x + 5 = 0$

Problems Involving Quadratics

In these problems be sure that you translate correctly the conditions of the problem into an equation.

1. The sum of two numbers is 12. The sum of their squares is 80. Find each number. (If one number is n, the other is $12 - n$.)

2. One number is 3 units larger than another. Their product is 54. Find the numbers.

3. The altitude of a triangle is 3 units larger than its base. The area is 9 square inches. Find the base and the altitude.

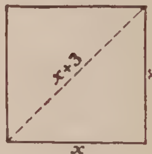
4. The square of a number is 56 more than the number itself. What is the number?

5. A number exceeds its square by $\frac{2}{9}$. Find the number.

6. For what values of n does $n^2 = n$?

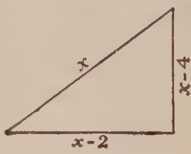
7. Show that the diagonal of any square is $\sqrt{2}$ times the length of its side.

8. The perimeter of a rectangle is 30 inches and its area is 54 square inches. What are its dimensions?



9. What is the length of a diagonal of a square if it is 3 inches longer than a side? (Use the Rule of Pythagoras.) Give the answer correct to hundredths.

10. A man wished to fit three rods together as shown in this illustration. The hypotenuse was to be 2 inches longer than the base and 4 inches longer than the altitude. How long should he make each rod? Use the Theorem of Pythagoras, thus: $(x-2)^2 + (x-4)^2 = x^2$.



11. By how much is the diagonal of a square whose side is s increased when the side is increased by 1? by 2? when the side is doubled?

12. By how much is the diagonal of a square decreased when the side is decreased by 1? by 2?

Largest or Smallest Value of a Second-Degree Expression

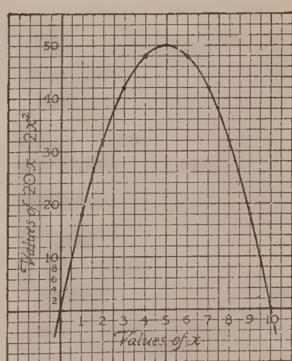
EXAMPLE 1. A rectangular garden is to be enclosed on three sides by fencing; the fourth side consists of the side of the barn. What is the largest garden that can be enclosed with 20 yards of fencing if the barn is on one of the longer sides of the rectangle?

Let x represent the number of units in the width.

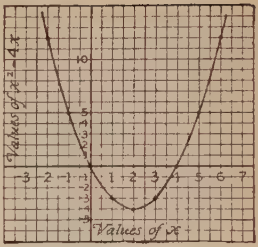
Then $20 - 2x$ represents the number of units in the length and $x(20 - 2x)$, or $20x - 2x^2$, represents the number of units in the area.

We now wish to know the greatest value of $20x - 2x^2$. This table shows how $20x - 2x^2$ changes as x changes.

When x is	0	1	2	3	4	5	6	7	8	9	10
$20x - 2x^2$ is .	0	18	32	42	48	50	48	42	32	18	0



If we plot the pairs of numbers in the table, we get the graph at the right. Note that as x increases from 0 to 5, the value of $20x - 2x^2$ increases and from that point on it decreases. From the graph we see that the highest point is (5, 50). The largest value of $20x - 2x^2$ is therefore 50 and the largest possible area is 50 square yards. The value of x (the width) which gives this value is 5 yards. The length is 10 yards.



EXAMPLE 2. Find the smallest value of the expression $x^2 - 4x$. The graph at the right shows how $x^2 - 4x$ changes with x. The lowest point of the graph is $(2, -4)$. The smallest value of $x^2 - 4x$ is therefore -4, and the corresponding value of x is 2.

Exercises

Find the largest or smallest value of the following expressions and the corresponding value of x:

1. $x^2 - 2x$
2. $x^2 + 4x$
3. $x^2 - 6x + 2$
4. $x^2 - 3x + 1$
5. $4x - x^2$
6. $2x - x^2$
7. $4 - 6x - x^2$
8. $2 + 3x - x^2$

9. A farmer's son wishes to fence a rectangular poultry yard whose area is to be 200 square feet. If one dimension is x, express algebraically the perimeter (the amount of fencing needed). Draw a curve showing how the perimeter changes as x changes. Look at the graph to see (1) what is the least amount of fencing required; (2) the dimensions of the chicken lot having the least fencing.

10. If you throw a ball upward with a velocity of v feet a second, the distance it will have traveled upward at the end of t seconds is shown by an experiment in science to be $vt - 16t^2$. Note that the distance it will have gone upward after a given time depends upon the velocity with which it is thrown and the number of seconds it has been traveling. Suppose you throw a ball upward with a velocity of 80 feet a second. The expression which tells you how far it has gone is $80t - 16t^2$. What is the greatest value of this number? In other words, how far will the ball go upward before it begins to drop? Make a graph of the expression $80t - 16t^2$. Now inspect the graph to see (1) how far the ball will rise, and (2) how long it will take the ball to reach the highest point.

Chapter Summary

In this chapter you learned how to solve second-degree equations with one unknown. These equations were of the form $ax^2 + bx + c = 0$, where a could be any constant other than zero, and b and c could be any numbers (constants) including zero. Equations of this type are *quadratic equations.* **Every quadratic equation has two roots.**

These equations are called *complete quadratic equations* when both the first and second powers of the unknown are present. They are called *incomplete quadratic equations* when the term containing the first power of the variable is missing.

It was possible to find the two roots of incomplete quadratic equations by the rules you already knew for solving other equations plus the use of two new axioms.

Complete quadratic equations of the form $ax^2 + bx + c = 0$ are solved by the use of the formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Since the application of this formula requires a clear understanding of positive and negative square roots, the early part of the chapter gave you information on this subject. A *square root* of a number is a number which multiplied by itself will give the number. Every positive number has two square roots, a positive one and a negative one. The positive square root of a number is called its *principal square root.* Mathematicians have agreed that when we speak of *the* square root of a number or when we use the radical sign, $\sqrt{}$, we shall mean the positive square root only.

Quadratics were found to have certain interesting applications, such as the Law of Pythagoras, that *the square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides.*

You should know the meaning of the following terms:

Square root
Principal square root
Quadratic equation
Incomplete quadratic equation
Complete quadratic equation
Root of an equation
Law of Pythagoras

Chapter Review

1. What is meant by the square root of a number?

2. How do you know that 6 is a square root of 36?

3. How many square roots does a number have? Explain.

4. Is $\sqrt{25}$ equal to -5? Explain.

5. Give the value of each of the following:

(a) $\sqrt{81}$ (c) $\pm\sqrt{.04}$ (e) $\pm\sqrt{\frac{81x^2y^4}{100z^6}}$

(b) $-\sqrt{36}$ (d) $\sqrt{16x^8}$

6. Find the square root of 5.362 to the nearest hundredth.

7. Using a table, find $\sqrt{42.5}$ to the nearest hundredth.

8. Compute $\sqrt{\frac{5}{8}}$ to the nearest hundredth.

9. If the hypotenuse of a right triangle is 9 inches and another side is 6 inches, how long is the third side (correct to hundredths)?

10. Give an example of an incomplete quadratic equation and also of a complete quadratic equation. Explain how to solve each.

11. In what form must a complete quadratic equation be in order to solve it by formula?

12. Solve $5x^2 - 7 = 2x^2 + 6$. Give the roots correct to the nearest hundredth.

13. What is the formula for solving a complete quadratic equation?

14. Solve $3x^2 - x = 4$.

15. Give the roots of $2x^2 - 3x - 1 = 0$ correct to the nearest hundredth.

16. What is the smallest possible value of the expression $x^2 - 7x$?

17. How high will a bullet rise if fired vertically with an initial velocity of 800 feet a second? Plot the number $800t - 16t^2$ and find (1) the greatest distance it will travel upward before it begins to drop, and (2) the time it takes it to rise.

CUMULATIVE REVIEW

1. In the equation $2(a + 3) = 2a + 6$, does a mean *any number* or *some number?* Answer the same question for $2a + 3 = 11$.

2. Translate each of these statements into clear, concise English statements:

(*a*) $a \times 0 = 0$
(*b*) $0 \div a = 0$
(*c*) $a + 0 = a$
(*d*) $1 \times a = a$
(*e*) $3(a + b) = 3a + 3b$
(*f*) $(a + b)^2 = a^2 + 2ab + b^2$

3. Complete the following statements concerning $abc - 2c^3$:

(*a*) The second term is _?_.
(*b*) The first term contains _?_ factors.
(*c*) The 3 in the second term is an _?_.
(*d*) The -2 in the second term is a _?_.
(*e*) Since the expression contains two terms, it is a _?_.

4. Give the rule for squaring the sum of two numbers.

5. If t represents the tens' digit of a two-digit number and u represents the units' digit, what is the value of the number?

6. What is the ratio of $7a$ to $8a$?

7. Express the ratio of 8 to 15 to the nearest thousandth.

8. If n is a positive number, which is the greater ratio,

$$\frac{2}{3} \text{ or } \frac{n+2}{3}? \quad \frac{2}{3} \text{ or } \frac{2}{n+3}?$$

9. Find the value of $2x + 3(5x - 1)$ when $x = -5\frac{1}{2}$. (Evaluate the parenthesis first.)

10. What is the value of $x^4 + x^3y - x^2y^2 - xy^3 + 2y^4$ when $x = -2$ and $y = 3$?

11. By inspection of the formula $A = lw$, tell what the effect is upon A if l remains constant and w is multiplied by 7. What is the effect upon A if both l and w are multiplied by 3?

12. By inspection of the formula $V = e^3$, tell what the effect is upon V if e is multiplied by 4.

13. Factor:

(a) $a^2b + ab^2$

(b) $x^3 - xy^2$

(c) $x^2 - x$

(d) $b^2 - 3b - 10$

(e) $\pi a^2 - \pi b^2$

(f) $3x^2 - 15x - 72$

14. If $y = 2x + 3$, express $x^2 - 3x + 1$ in terms of y.

15. What algebraic expression represents the increase in the area of a square whose side is x, if you increase its side by 4 units?

16. Reduce the following fractions:

(a) $\dfrac{a^2 - b^2}{2a + 2b}$

(b) $\dfrac{3n^2 - mn}{3mn + n^2}$

17. Solve graphically: $x - y = 7$
$3x - y = -17$

18. Perform the indicated operations:

(a) $20x^6y^4 \div 4x^2y^3$

(b) $(3a^2 - 6ab) \div 2a$

(c) $(3x^3 - 6x^2 + 1) \div (x - 3)$

(d) $(x^2 - 5xy + 2y^2) - (3x^2 + 2xy - 7y^2)$

(e) $\dfrac{x^2 - y^2}{x - 2y} \times \dfrac{x^2 - 5xy + 6y^2}{5x + 5y} \times \dfrac{1}{x - y}$

(f) $\dfrac{3a - 5b}{3} + \dfrac{a + 2b}{4} - \dfrac{5a - 7b}{6}$

19. Solve the following equations:

(a) $n + 4.5 = 9.2$

(b) $3n - 2.3 = 1.3$

(c) $\frac{1}{5}a + 7 = 9$

(d) $8x - 7 - x - 11 = 3x - 6$

(e) $4x - (2 - x) = 3x + (3x - 1)$

(f) $(3x + 4)^2 - 9x(x + 2) = 4$

(g) $(2x + 3)(2x - 3) - (x - 2)^2 = 51$

20. Solve for c: $a = b(c + d + 4e)$

21. Solve for n: $S - \frac{na}{2} = \frac{nl}{2}$

22. Solve for v: $F = \frac{mv^2}{R}$

23. Solve for x and y:

(*a*) $7x + 9y = 8$
$9x - 8y = 69$

(*b*) $.2x - .3y = 2.6$
$.2y - .5x = -3.2$

(*c*) $\frac{x+6}{3} + \frac{y-2}{2} = 4$
$\frac{4}{3}x - 5 = \frac{y-6}{2}$

24. Write the formula for the area of a triangle. Compute the area to the nearest square foot of a triangle whose base is 23.9 ft. and whose height is 8.7 ft.

25. Write a formula for the cost (C) of painting a ceiling a ft. by b ft. at 18 cents a square yard.

26. One group of airplanes leave their airport at 6 P.M. and expect to reach their objective at 9 P.M., flying at 250 miles an hour. Another group leave the same airport at 7 P.M. What must be their rate of flight to reach the same objective at 9 : 30 P.M.?

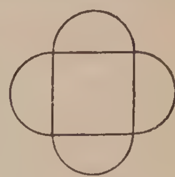
27. The base of a triangle exceeds its height (h) by 10 in. Express the area in terms of h.

28. What per cent of 75 is 24 (to the nearest tenth)?

29. 75 is 24% of what number (to the nearest tenth)?

30. How long will it take \$2500 to earn \$412.50 when it is invested at 3% simple interest?

31. This figure consists of a square and four semicircles. If each side of the square is $2r$, what is the perimeter of the figure in terms of r? What is the area of the figure in terms of r?



32. The entire area of a cylinder is given by the formula $A = 2\pi r(r + h)$, where r is the radius and h is the height. Find the area when $r = 3.5$ and $h = 8$.

33. The metal in a ball with a 4-inch radius is to be used to make balls which have a 1-inch radius. How many balls can be made, assuming no waste? (The formula for the volume of a ball is $V = \frac{4}{3}\pi r^3$.)

34. The side of a square room is 21.5 feet. Find the length of a diagonal of the floor correct to tenths.

35. What is the selling price of a suit that cost \$30 and is sold at a margin of 40% of the selling price?

36. John has \$10.50 in a savings account and is depositing 50 cents a week. Ella has \$15.25 and is depositing 25 cents a week. In how many weeks will John have as much as Ella, neglecting interest?

37. One airplane leaves A at 8 o'clock and flies toward B at the rate of 250 miles an hour. Another plane leaves B at 9 o'clock and flies toward A at the rate of 300 miles an hour. If A and B are 2175 miles apart, at what time will they meet?

38. A searchlight on the top of a building is 180 ft. above the street level. Through how many degrees from the horizontal must its beam of light be depressed so that it may fall directly on an object 400 ft. down the street from the base of the building?

39. A plane starts at O and flies in a direction which is 24° west of north, at the rate of 315 miles an hour. In two hours, how far north is it of an east-west line through O? How far west is it from a north-south line through O?

NOTES TO THE TEACHER

NOTE 1 (*Page 1*).[1] Algebra is essentially a symbolism with accompanying rules of procedure to help men think through the quantitative relationships which they meet. It differs from arithmetic in that the numbers are general or unknown. The letters are not shorthand symbols. They are numbers and are operated on just as the numbers of arithmetic are operated on.

C.O.D. is not algebra. $A = lw$ is algebra. But when a pupil sees this formula for the first time, he has the same attitude toward it, if we are not careful, as he does toward C.O.D. To him, A is an abbreviation for area, l for length, and w for width. The formula appears to be merely a short way of writing the longer English rule for finding the area of a rectangle. His first impression is wrong and he may go through the whole course believing that he is juggling letters according to given rules.

We can introduce algebra through generalized arithmetic so that the right impression is given from the start. An opportunity is given to think about the processes in arithmetic already learned and to generalize them. The generalized statements are understood because they are seen through a background of specific arithmetic statements. Then, when the statements are made in algebraic symbols, the pupil does not see the statements standing alone, merely letters with symbols of operation, but he sees them as a simple way of collecting into one statement all the specific cases. To him $a + b = b + a$ means $5 + 4 = 4 + 5$ and $6 + 2 = 2 + 6$ and $10 + 3 = 3 + 10$, and so on ad infinitum. The letters *are* numbers. The first impression is important.

In this approach pupils should first learn the nature of written symbols. They are nothing but marks which have been given meaning. Then they will be ready for new meanings put into the letters with which they are so familiar in reading.

Chapter I is written on the assumption that the first impression the student should have of algebra is that it is generalized arithmetic. He should understand from the start that the letters are numbers. He is going one step further than his arithmetic. He is dealing with familiar relationships, but generalizing them by means of a new symbolism. At no time should he be allowed to get the impression that he is working with letters which are abbreviations.

Merely to tell the pupil this is not enough. Understanding must grow on him. He must have varied experience with letters as general numbers in order to establish the concept. The approach should be inductive so that when he sees a general statement in algebraic symbols, he will think of it as meaning all the specific statements he has seen put into one statement.

We begin by developing the meaning of such statements as $a \times 0 = 0$, $1 \times a = a$, $a + b = b + a$, $a - a = 0$, and $ab = ba$. These introduce the idea of letters as general numbers. Then we develop the method of combining like terms by generalizing from arithmetic. This tends to show that the rules of algebra are not arbitrary. While the first step introduces the idea that the letters are numbers, this second step begins to establish the idea. It avoids the notion that $2a + 3a = 5a$ just as 2 apples plus 3 apples

[1] The page numbers in parentheses indicate the pages of the text where reference to the different notes are made.

are 5 apples, a poor psychological approach. $2a + 3a = 5a$ because they have seen it work with several specific numbers.

We then turn to checking subtraction and division, not to show the pupils how to check subtraction and division — they already know how — but to let them see once again in a very familiar situation how letters are used as generalized numbers. Experience of several teachers using the text before publication shows that by now pupils are beginning to use the symbolism with some confidence. With many pupils it will be possible to go immediately to the general case without first giving specific instances. Writing $\begin{array}{r} a \\ \underline{b} \\ c \end{array}$ on the board and asking, "If I subtract b from a and get c, how will I check the subtraction?" will bring the response at once, "Add c to b to see if you get a." That is the purpose of this first chapter — to start the pupil thinking more generally about number relationships with which they are already familiar and to develop their ability to express the general statements by means of letters which they think of as numbers.

NOTE 2 (*Pages 3, 6, 317*). One disadvantage a textbook has in contrast with oral teaching is that it cannot ask a question and get the reaction of pupils before it asks another. When the two questions are written, one after the other, the second question may give away the answer to the first and deprive the pupils of the value of thinking through the first question. For this reason, teachers are urged to become familiar with the developments and key questions on pages indicated by a dagger (†) and to discuss these pages with the class while the pupils have their books closed.

Since it is advisable for the purpose of establishing meanings once achieved to have answers to the questions available, the answers are printed in the book in parentheses. After the oral development, the pupils may read the pages to set the ideas more firmly.

NOTE 3 (*Pages 12, 123*). If pupils have difficulty in getting the answers to a general problem involving literal numbers, they will be helped by rereading the problem and replacing the literal numbers by simple arithmetical numbers. For example, Ex. 4, page 13, may be read: How many minutes are there in 2 hours? 3 hours? 4 hours? The answers to these questions will indicate that a is to be multiplied by 60 and that $60a$ is the answer to Ex. 4. Pupils will have to be guided in doing this by the teacher until they have formed the habit for themselves.

NOTE 4 (*Page 18*). Pupils who write, "If $n + 4 = 8$, $n = 12$," or "If $\frac{x}{2} = 8$, $x = 4$," as many do at the beginning of their study of algebra, have no clear understanding of the meaning of an equation. It would seem that they are so distracted by trying to get an answer according to the rules that they forget that the purpose of solving an equation is to find a number that will satisfy it, and so they give absurd answers without being disturbed. The advantage of using trial and error for a brief period in solving equations is that the attention is focused upon the essential meaning of a solution — a number which will satisfy the equation. Since the only operation that pupils perform is the check, its importance in connection with the solution of equations is emphasized.

NOTE 5 (*Page 24*). A chapter review is given for each chapter to make it easier for both teacher and pupil. These reviews may be given as written or oral tests or may be assigned for a home lesson. The results may be used to diagnose pupils' difficulties and to indicate what parts of the chapter, if any, need further practice.

In addition to these reviews the authors provide a set of chapter tests, bound separately, which may be used for grading purposes.

NOTE 6 (*Page 27*). Chapter I has given the pupil confidence in the use of letters as numbers. He is ready now to see how this symbolism can be used. Its use is shown in four of its most important phases: (1) direct use of the formula, (2) finding unknowns by solving equations, (3) indirect use of the formula, and (4) problem solving.

The direct use of the formula — the case where the given formula is already solved for the desired letter — is one of the most important uses of algebra and the one that requires the least knowledge of it. It requires only an understanding of the meaning of its symbolism and the ability to carry out the indicated arithmetic operations. This use of algebra is illustrated in this chapter by means of the Fahrenheit-centigrade formula. Incidentally the parenthesis is introduced and practice given with its use.

Finding unknowns by means of the rules for solving equations is at the core of all work in algebra, as we have said in our introductory paragraphs. In Chapter I, pupils learned the meaning of an equation. Now they are shown definite methods of solving equations. The rules of algebra take away all the guesswork.

The power of algebra is shown in connection with our third illustration, the indirect use of the formula. In this case the formula is not already solved for the desired letter. After substituting the given values, it is necessary to solve an equation to find what is called for. Algebra makes these indirect examples easy, as is shown by its application to the second and third cases in percentage.

Our fourth illustration of the use of algebra is in problem solving. Concerning this we have written a separate note. (See Note 9.) Certain other topics have been introduced in this chapter because they are needed for the understanding of the four topics which we have discussed.

NOTE 7 (*Page 34*). Teaching the solution of all four types of simple linear equations at the same time is preferable to teaching the solution of each type separately. In the latter case, pupils learn by imitation, largely without understanding. Then, when they are given a miscellaneous set of exercises, they are confused. When taught in the former way, they have the proper mind-set from the beginning. They learn from the start to make a proper choice among addition, subtraction, multiplication, and division cases.

Understanding of the inverse processes and their use in the solution of equations is important. Recognition of what has been done to the unknown in the given equation automatically determines the method of solution of the equation.

NOTE 8 (*Page 48*). You should not let the pupils lose sight of the fact that the letters of algebra are numbers. Consider Ex. 1, for example. $2(a+b) = 2a + 2b$. You should ask, "Are a and b any numbers or are they some particular numbers?" Then let the pupils take any values they choose for a and b and see if the statement is true. In addition, have the pupils make the statement in English; in this case, "Twice the sum of two numbers is equal to twice the first number plus twice the second number." This sort of thing should be done at appropriate intervals throughout the course.

NOTE 9 (*Page 52*). The method of problem solving given in this book has been found to obviate many of the usual difficulties. Pupils like it because of its direct numerical approach to the analysis. By taking an answer at random and then checking to see if it is right, they learn the various relationships in the problem. Working back from a numerical answer to the conditions of a problem is a direct arithmetic procedure and more within the pupils' experience. Once the processes and relationships are discovered in this direct way, it is simple to use an unknown instead of the numerical guess and then proceed algebraically with exactly the same processes as were used in the arithmetic check. Guessing the correct answer, of course, is not important.

Mastery of problem solving does not come in minutes or even in a few days. You should expect the tempo of progress to be slow and teach accordingly. Until a class shows signs of a clear grasp of the method, problems should be done in class with the teacher and pupil working together. Home assignments should not be given until the pupils have a feeling of confidence.

The aim should be the learning of a general method of analysis in problem solving rather than particular methods for special types of problem.

NOTE 10 (*Page 109*). It has become increasingly clear that high school students and college students as well do not have an adequate mastery of arithmetic. For this reason, several pages of drill in fundamental operations with integers, fractions, and decimals have been provided.

Arithmetic is also introduced in connection with the algebra throughout the course. The skills of arithmetic should not be neglected in the high school.

NOTE 11 (*Page 117*). The interpretation and construction of graphs becomes more and more important in the solution of problems. It is particularly important at this time because many of the problems of navigation are solved by means of graphing.

The fundamental ideas in the graphing of a formula or an equation are that a number can be represented by a length of line and a number-pair by a point at proper distances from reference lines. These ideas are carefully developed here and should be stressed by the teacher.

NOTE 12 (*Page 133*). Rules for operating with signed numbers should come only after their meaning is clear. Avoid the idea that a negative number expresses a quantity which is less than nothing. This is a misconception which causes a great deal of difficulty. How can a man walk -5 miles — 5 miles less than nothing! Stress the correct concept that positive and negative numbers indicate oppositeness in quality.

NOTE 13 (*Page 143*). Rules for the addition and subtraction of signed numbers have little meaning unless they are developed inductively. To say to pupils, without any preparation, "To *add* a positive and a negative number, *subtract* . . ." sounds queer, to say the least. Once the meaning of addition has been enlarged to include the addition of negative numbers, pupils can find the sums of signed numbers by means of a number scale, and with the help of the teacher develop the rules for addition. The meaning of subtraction should be made clear so that pupils can first subtract numbers by means of the number scale. The rule which can then be developed merely gives them a simpler and easier way to get a result they already understand.

Throughout the chapter, use every opportunity to emphasize the fact that the letters represent numbers. Pupils should have constantly in mind that they are not juggling with letters. They should realize that they are working with actual number relationships such as they learned in particular in arithmetic.

Rules for the removal of parentheses should be definitely connected with the rules for addition and subtraction of signed numbers.

NOTE 14 (*Page 152*). We continue the process of showing the pupils that the algebraic processes are generalizations of relationships which they can readily see in connection with arithmetic numbers.

NOTE 15 (*Page 171*). Equations are introduced here and at other places throughout the text for two reasons: (1) to show the use of principles just learned, and (2) to avoid loss of skill in solving equations.

NOTE 16 (*Page 179*). Be particularly careful in discussing the examples in this section. It is in this kind of work with signed numbers that pupils find the greatest difficulty. Careful discussion of the errors made in the set of exercises on page 183 will help to remedy misunderstandings. It may be advisable to repeat exercises of this type from time to time until the ideas are thoroughly mastered. More difficult exercises, many of them involving fractions, are found on pages 183 and 184.

NOTE 17 (*Page 216*). It is easy for pupils to forget that the letters of algebra represent numbers after they have spent several days working with the fundamental operations with literal numbers. This is, therefore, a good time to review the meaning of algebraic symbolism. Note carefully the three things pupils are expected to do with each of the exercises. *Practice in meanings should go hand in hand with practice in skills.*

NOTE 18 (*Pages 221, 308*). Here are several pages of review exercises. They are placed here for your convenience in keeping up a constant review. It may be well to spend two days definitely on this review, but you cannot cover the whole set of exercises in that time. After that you can assign a few exercises as a part of each day's home lesson as you continue with new work.

NOTE 19 (*Page 277*). You should make at least one graph, working with the pupils. While you do the work on the board, the pupils should follow you step by step on paper. Unless this is done, pupils may miss some of the details. Only three points need be plotted for each equation, two for the line and one for the check.

NOTE 20 (*Page 343*). As stated in the text, the aim of the study of fractions in algebra is twofold: (1) to give increased understanding and skill with arithmetic fractions, and (2) to develop understanding and skill with literal fractions to be used in future work. The first aim should not be slighted.

General concepts should be introduced by means of specific examples in arithmetic. Concepts should be established by means of practice not only with literal but arithmetic fractions. The more careful analysis of principles needed to work with literal fractions will aid in a clearer understanding of the processes with arithmetic fractions.

Most of the work in fractions is dependent upon an understanding of equivalent fractions. It is wise to make sure that students see intuitively the possibility of a fraction having many forms such as $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, etc., before the more abstract rules are given. Simple examples, such as "One half pound of butter is the same as two quarter pounds," will help. Even though this idea has been before the pupils since the early grades, it can never be made too clear. Once the rules for changing a fraction to an equivalent fraction are given and pupils are using them with literal fractions, it is wise occasionally to substitute numbers for the letters, thus getting equivalent arithmetic fractions. This will help to make the abstract work more understandable.

The amount of work done in fractions should vary according to the ability of the students. For slow pupils, fractions may be restricted to those with monomial denominators. Complex fractions may be omitted. A good rule to follow with such pupils is this: "No more difficult operations with fractions or fractional equations need be taught than those that are necessary for evaluating commonly used formulas." The better pupils are most likely to be the ones who will take further courses in mathematics. They need a good foundation in fractions.

NOTE 21 (*Page 347*). Frequent errors in fractions are due to the cancellation, so called, of like expressions from numerator and denominator. Even when pupils are clear on the point that the reduction of a fraction depends upon *dividing* the numerator and denominator by the same number and not upon any other operation, they may persist in absurd "cancellations." Often enough the difficulty is that the pupils think they are dividing whenever they draw a line through an expression. If pupils can be led to analyze what they are doing instead of crossing off like expressions indiscriminately, most of these errors can be obviated. The purpose of this section is to lead the pupils to this analysis.

NOTE 22 (*Page 357*). A student's first reaction when given an example in the addition or subtraction of fractions should be, "I cannot combine them unless they have the same denominator." If this is clear, there are then two things he must know in order to continue: (1) how to add or subtract fractions when the denominators are the same, and (2) how to change them to the same denominator when they are different.

In this section ample practice is given in adding fractions having the same denominator. Note that various difficulties which may be met in adding numerators are taken care of in this set of exercises, so that they can be mastered before the student has the added difficulty of changing to a common denominator.

INDEX

IT'S SMART TO BE CAREFUL

1. Look up and down the street before crossing - STOP-LOOK and LIVE.
2. Do not walk between parked cars.
3. Be alert - don't get hurt.
4. Do not cross the street against the light.
5. Obey all traffic signs and rules.

UNIVERSAL SAFETY